This is not the title of this theme

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1 Paradoxes – here, there, everywhere

I am very sorry - the great French mathematician and philosopher Blaise Pascal wrote to one of his friends - *but I don't have enough time to make my letter shorter...* At first glance this thought might seem paradoxical but after a while we realize how difficult it is *to say in ten sentences what other men say in whole books,* and even more so - what other men do NOT say in whole books... What is *paradox,* by the way? Even the dictionaries disagree on this - according to Webster the paradox is *a statement which though true seems false and self-contradictory* whereas Longman defines it *as a statement which seems to be foolish or impossible but which has some truth in it.* The word stems from the ancient Greek where it had the meaning "contrary to the accepted opinion".

One of the best mathematics popularizers, Martin Gardner, classifies the paradoxes in four main categories [1]:

- Assertions that seem false but actually are true
- Assertions that seem true but actually are false
- Lines of reasoning that seem impeccable but which lead to logical contradiction (more commonly called a *fallacy*).
- Assertions whose truth or falsity is undecidable.

Life is full of paradoxes (in any of the above senses) to the extent that it would be paradoxical not to come across paradoxes every now and then. A couple of weeks ago I asked in a bookshop for *What is the name of this book?* - a famous book on logical paradoxes by Raymond Smullyan. The book seller got very angry: "You don't know the name and you expect ME to know it..."

The next day a fax arrived whose only readable sentence read: "If you don't receive this fax, please call us..." (A nice variation of this would be: *I have not received the letter in which you are reminding me about the money I owe you...)*

Some of you have probably come across advertisements of the kind: IF YOU CAN'T READ THIS, CALL 898-60-01. Or have seen (and even suffered from) falling wooden inscriptions of the kind: WATCH OUT - FALLING OBJECTS! Others might have even written on a wall as in Fig. 1:

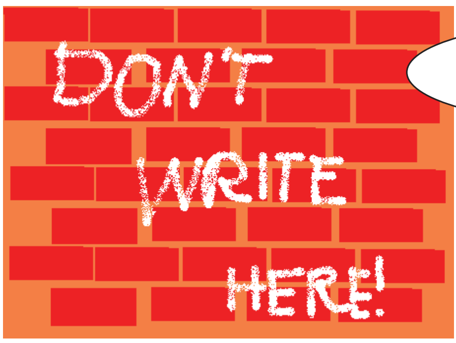

Fig.1 A paradox we often see on the walls.

Fig.2 *Cretans, always liars*. (Artist: Yovko Kolarov)

Here follow a couple of stories which, even if not true, might be such:

- At a lecture in mathematics someone from the audience told the lecturer: *There is an error on the black board.* The answer was: *I have been wrong only once in my life – when I thought I was wrong...*
- Bertand Russel asserted that George Edward Moore (a distinguished English philosopher) had told a lie only once in his life. This was when he was asked if he had always said the truth. Then Moore thought for a while and said: *No.*

Many aphorisms, jokes and cartoons are based on paradoxes. Here are some examples.

Oscar Wilde, the great Irish playwright, said: *Paradoxically though it may seem, it is none the less true that life imitates art far more than art imitates life.*

Grucho Marx, a master of wit, is remembered by his paradox-like one liners:

- *I don't care to belong to a club that accepts people like me as members.*
- *I find television very educating. Every time somebody turns on the set, I go into the other room and read a book.*

Stanisław Jerzy Lec, one of the most influential aphorists in the 20th century, included in his *Unkempt Thoughts* the following:

- *If a man who cannot count finds a four-leaf clover, is he lucky?*
- *The radio is a wonderful invention – you just push the button and there will be silence...*

In one of the cartoons about the Peanuts, Charlie Brown shares with Lucy that he has a new life philosophy – to say NO to each question.

– *To say NO to each question?* – asks Lucy unbelievingly.

– *Yes, I mean NO,,, O-o-oh, you ruined my new life philosophy – sighs Charlie.*

1.1 Tasks and problems

Task 1. Think of a paradox you have come across in your everyday experience.

Task 2. Try to translate in a language which is not English (e.g. German) the following sentence: *This sentence in English is difficult to translate in German.*

Task 3. Here follows a fragment of a list of rules compiled by Harold Evans as quoted in [1]. Explain where the paradox lies in each one of them:

- *Don't use no double negatives.*
- *Don't use commas, which aren't necessary.*
- *Verbs has to agree with their subjects.*
- *About those sentence fragments.*
- *Try to not ever split infinitives.*
- *Always read what you have written to see you any words out.*
- *It is important to use apostrophe's correctly.*

Task 4. Try to enrich this list with paradoxical grammar rules of your own.

Task 5. Think of (or find) a paradoxical title of a book, article, textbook.

Task 6. A grandmother pulled a wishbone with her small grandson. Her wish was that *he* would win. She pulled the larger part of the bone. Did she win? Explain your reasoning.

2 *Is this a paradox or isn't it?* – this is the question

Of course, what appears to be a paradox is often insufficient mathematical knowledge or even absent-mindedness. Just remember the boy who having learned how to work with a calculator, exclaimed in surprise: 1/2 *is a small number,* 1/4 *is also a small number, but look at how big* 1/3 *is!!!* (What was his mistake?) Or, the teacher who complained: *The larger half of my students doesn't know that the two halves of anything are equal...* And the absent-minded professor who said: *There are three types of mathematicians - those who can count, and those who can't.*

Even the famous paradox about the liar [2] could hardly be called *a paradox*. Let us recall the story – it is about Epimenides, a poet of 6th c. BC, from Crete, who according to the legend had said: *Cretans,* *always liars* (Fig. 2). It was considered for a long time that this statement of Epimenides was a paradox. Here is how the ancient Greeks reasoned (try to find the flaw in their reasoning):

> *If the assertion is true than it is false (since all the Cretans are liars and therefore Epimenides who is Cretan himself, also lies. If the assertion on the other hand is false, then all the Cretans tell the truth which would mean that it is true – again a contradiction.*

Did you find the error in these reasoning? Of course – the logical negation of the assertion *All the Cretans are liars* is not *All the Cretans tell the truth* but rather *Not all the Cretans are liars.* “Not all” (or “not every”) does not mean the same thing as “none”; it translates as “some” [3], i.e. the logical negation in fact is *There exists at least one Cretan who tells the truth.* This means that the paradox is not a real paradox – Epimenides is a liar but there are some Cretans (at least one) who tell the truth.

By the way, it is not always easy to make the negation of a statement. When we form the negation of a proposition which is true we expect its negation to be false. But consider the following examples;

- *This sentence consists of six words.*
- *Half of the students in the class are stupid.*

Sometimes a non-paradox becomes a paradox because words change their meaning while the phrases containing those words remain intact. How many times have you heard the phrase: "The exception proves the rule" used to argue that a rule is all the stronger and more meaningful when you can point out times it has been broken? The phrase, however, uses the word *prove* in its older meaning of *test.* You would come closer to its original meaning of the phrase by using a synonym and saying, "The exception probes the rule", just as you would verify if a computer program is working properly... *Understood correctly, the original saying affirms rather than denies the most basic of principles of scientific inquiry: that one must account for exceptions, not ignore them* [4].

2.1 Tasks and problems

Task 7. Suppose you say: *I am lying.* Is what you say true or false*?* Explain why this is a paradox.

Task 8. Lucy asserts that the negation of $x<10$ is $x>10$. Do you agree? Explain.

Task 9. Try to negate the following sentences:

- *Everybody in this room is sleeping.*
- *Nobody pays attention.*
- *Some students have solved the problem.*
- *All the students in this class are girls.*
- *There is at least one person who has read up to here.*
- *Not all the problems are challenging.*

3 Some paradoxes in science

Although strange, the paradoxes are an essential part of the *great science* because it is looking for "crazy", i.e., paradoxical theories. This is well illustrated by Nils Bohr whose reaction to the paper of Pauli and Heisenberg was: *We all agree that your theory is insane. The question is if it is really that insane so as to have the chance to be true...* No wonder that science goes forward in harmony with the number and the depth of the paradoxes overcome, with the paradoxality of the new ideas. Of course science does not stay at one spot. Very often something that seems to be a paradox soon is accepted as quite normal. Take for example the following ex-paradoxes:

- *In vacuum heavy objects do not fall faster than the lighter ones.*
- *Heat is movement.*
- *Malaria is caused by the mosquitoes.*
- *The odd numbers are as many as the natural numbers.*

What has happened with the last paradox, and many other of the type “the whole equals the part”, has now become the defining property of the *infinite*.

Even one of the greatest achievements of our century, the theory of relativity, was commented by Rutherford as nonsense (*We don't need this for our work...).* At the time Rutherford was already a world-famous scientist and this theory was not a novelty.

The relativity of time is illustrated by the Twin paradox - if one of the twins is launched in a rocket he could return to Earth in his own future. The time on the rocket will run slower because of the greater speed and when the twins meet, the longer the travel and the greater -the speed of the rocket have been - the bigger their age difference will be (Fig. 3). As Hardy says: *If it were not for Einstein, the physical picture of the world would be different...*

Fig.3 The Twin paradox after Einstein.

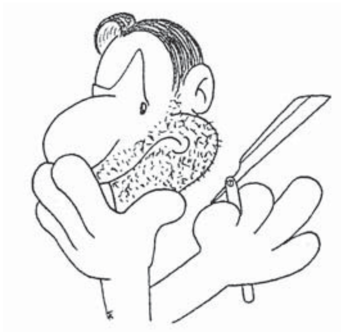

Fig.4 *If I can shave only those who can't shave themselves, can I shave myself*?

(Artist: Yovko Kolarov)

But maybe the greatest paradox in scientific context is that in mathematics there are also paradoxes which can lead to deep insights.

3. 1 Paradoxes in mathematics

During its history mathematics has experienced three crises accompanied with revealing paradoxes. But solving them gave rise to new theories. The first crisis was provoked by the insight that the diagonal of the square and its side cannot be measured with one and the same unit. Overcoming this antique crisis gave rise to the irrational numbers (even their name shows how far from the "rational" they have seemed to the old Greeks - they even did not interpret them as numbers but rather translated them in terms of geometry.)

The second crisis caused a lot of problems to the mathematicians of 17^{th} and 18^{th} centuries in connection with the infinitesimal. The crisis due to the fact that these magnitudes have been treated both as 0 and non-zero was solved by Cauchy by introducing the theory of limits.

The last (for now) crisis (19th - 20th century) was so strong as to influence the foundations of mathematics. The eminent German logician Frege had just completed *The Fundamentals of Arithmetic*, in which he believed he had developed a consistent theory of sets that would serve as the foundation of all mathematics. The volume was about to be printed when Frege received a letter from Russell telling him about a paradox he had discovered about sets: *Consider the set of all sets that are not members of themselves. Is it a member of itself?* However you answer you will contradict yourself [1]. To make this more understandable to the general public Russell *dressed* the paradox in what is now known as the *Barber paradox*: *If a barber is allowed to shave all and only those men who can't shave themselves, can he shave himself?* (Fig. 4). Frege's set theory permitted the existence of the set of all sets which are not members of themselves. As it became clear from Russell's letter such a set is self-contradictory. Of course Frege was not very happy (to say the least) but according to Russell *thanks to revealing and overcoming the paradoxes mathematics became more logical and the logic - more mathematical...*

In *his theory of types* Russell made a *resolute attempt* to find a solution to the paradoxes. He arranged sets in a hierarchy of types in such a way that it is not permissible to say that a set is a member of itself, or not a member of itself. In such a way he eliminated self-contradictory sets. There is no meaningful way to define them if obeying the rules of the *theory of types* [1].

In short, any time when mathematics experiences a serious crisis it was saved by a new idea which re-gained its image of *unmistakable* science. That is why we should not be afraid of paradoxes but should even be searching for some which might blossom in beautiful theories.

One of the most impressive and the most complicated paradoxes in mathematics is Kurt Gödel's famous *Incompleteness Theorem*. It can be thought of as arising from his attempt to replicate as closely as possible the Epimenides paradox in purely mathematical terms. Let us consider the sentence: *This proposition can never be proven*. If it is false, it follows that it could be proven, which means that it is true and therefore – unprovable. The paradox is due to the fact that the notion *provable* is not well defined. In mathematical logic one doesn't talk about proof in an absolute sense but rather about *provability in a specific system.* Gödel shows that in any mathematically powerful system *S* it is possible to express a statement that closely reminds the liar paradox: *This formula is unprovable within axiomatic system S*. More precisely, Gödel manages to construct such a proposition that neither it, nor its negation could be proven in the given system.

I apologize to the readers for not having enough time to make this theme shorter. If you, by any chance, have read up to here, be sure that this is not its last sentence...Or is it?

P. S. When I asked a colleague of mine how he found this "might-be-a-Math2Earth theme", he murmured something like: *It could be worse...* My reaction was obvious: "Is this all you can say?" *Not really, I could have said: "It could not be worse"...*

Now it is time to check how logical you have become ☺

3. 2 Tasks and problems

Task 10. Think of variations of the Barber paradox (e.g. replace the barber with a cook, a painter, a robot).

Task 11. There are many numbers that could be called *interesting* because of their properties (e.g. 7 is *prime*, 28 is *perfect*, etc.)

- Do you think that every natural number is *interesting*? Try to argue by assuming the opposite. What can you say about the smallest number which is not interesting?
- Modify your argument in the case of *interesting* and *dull* people. Are there *dull* people according to your argument?

Task 12. Read the following dialog carefully:
The teacher: Who tore this book?
Becky: *I didn't!*
Tom: *I did!*
Joe: *Only one of them is telling the truth!*

Is it possible that Joe tells the truth if we know that just one, Tom or Becky, has done this?

Task 13. There are 3 false statements here:
- 2+2=4
- 3x6 = 17
- 8/4 = 2
- 13-6=5
- 5+4 = 9

Which are they?

Task 14. Is the following dialog a paradoxical one?
Plato: *What Socrates will say now will be a lie!*
Socrates: *Plato tells the truth!*

Task 15. Explain why the letter O has been replaced by a hexagon in the following sentence [5]:

If π = 3 this sentence w○○uld l○○k like this

Task 16. Try to construct sentences which *talk about themselves* [6, 7], like these:

- *In this sentence, the number of occurrences of 0 is 1, of 1 is 11, of 2 is 2, of 3 is 1, of 4 is 1, of 5 is 1, of 6 is 1, of 7 is 1, of 8 is 1, and of 9 is 1.*
- *This point is well taken*

Task 17. Write on one side of a blank card the following text:

THE SENTENCE ON THE OTHER SIDE OF THIS CARD IS TRUE.

On the opposite side of the card write:

THE SENTENCE ON THE OTHER SIDE OF THIS CARD IS FALSE.

Show the card to your friends and ask them which sentence is true.

References

[1] Gardner, M. *Aha! Gotcha – paradoxes to puzzle and delight*, W.H. Freeman and Company, New York, 1995
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[4] Goldenberg, E. P., Feurzeig, W. *Exploring Language with Logo*, The MIT Press, 1987, p. 170
[5] Hofstadter, D.R. *Metamagical Themas: Questing for the Essence of Mind and Pattern*, Basic books, 1985
[6] Burbanks, A. *Self-referential sentences* http://lboro.ac.uk/departments/ma/gallery/selfref/index.html, [accessed 18.01.2010]
[7] http://www2.vo.lu/homepages/phahn/humor/self_ref.htm [accessed 18.01.2010]

Recommended further readings

(Readings suggested by Martin Gardner in [1] are preceded by *.)

Byers, W. *How Mathematicians Think*, (*Using Ambiguity, Contradiction, and Paradox to Create Mathematics*), Princeton University Press, 2007

Gardner, M. *The Paradox of the Unexpected Hanging*, Chapter 1 in *The Unexpected Hanging and Other Mathematical Diversions*, New York: Simon&Schuster, 1968

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