Optimizing under constraints

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1 Introduction

How do computers solve complicated problems in so many different fields as economics, engineering, management, cosmology, etc.? Usually, the answer is: there exists some software to do the job. But is this really so? You can not write a separate piece of software for each and every application in existence simply because it would be prohibitively labour-consuming and unprofitable. It is here that **Mathematics** comes to help. Mathematics describes real-life problems in its formal language by finding what is common in them. It extracts the essence, places these problems in groups (**classes**) and gives **methods** to solve them independently of their real nature.

The process of translation of a real-life problem to mathematical language is known as **mathematical modeling**. The corresponding mathematical description is called a **mathematical model** of the real-life problem. The model obtained belongs in turn to a class of mathematical problems, for whose solving there already exist (or, are yet to be created) common methods . The solving itself is done by a software which performs these methods. In other words, Mathematical modeling serves as a bridge between the real-life problems and the software that solves them.

The mathematical modeling is done by various specialists (depending on the nature of the real-life problem) who, in the process of modeling, decide which features of the real-life problem to include in the mathematical model and which ones to omit. This is essentially a trial-and-error process. Once a mathematical model is created, it is solved by means of a software and the results are analyzed. If they are unacceptable, the mathematical model is refined by including new details and/or excluding old ones. This process may be repeated many times until the model becomes sufficiently close to the real-life problem. However, no matter how detailed the mathematical model is, it cannot entirely describe a complex real-life problem.

Here we will discuss a class of mathematical problems known as **Linear Programming** (LP). The word ”programming” in Linear Programming is completely different from ”programming” in Computer Programming. In the latter case, it means writing code for performing calculations. In LP programming means to plan and organize some activities in an optimal way. The LP problems are **optimization problems**. Any optimization problem has a **goal** to be reached under some **constraints**. The term *linear optimization* is often used as a synonym for linear programming.

LP problems were first formulated and solved in the late 1940’s. Now, Linear Programming is applied to various fields: business and economics, engineering, transportation, energy, telecommunications, manufacturing, planning, routing, scheduling, etc.

We consider below two simple examples of real-life problems, whose mathematical models are LP problems and we propose **graphical method** to solve them.

2 Two simple models of LP problems

The Product Mix Problem A company produces regular and super gasoline using three different grades R_1, R_2, R_3 of crude oil. The profit obtainable from each unit of the regular and the super gasoline is \$1 and \$4, respectively. The amounts of resources R_1, R_2 and R_3 available are 24, 10 and 30 units, respectively. The production of one unit of the regular gasoline uses 6 units of R_1 and 6 units of R_3. The production of one unit of the super gasoline uses 4 units of R_1, 5 units of R_2 and 10 units of R_3. The problem facing the company is to determine the amounts of regular and super gasoline (the *product mix*) that maximize the profit.

Notice that this real-life problem is an optimization problem. Our objective is to maximize the company’s profit. The constraints come from the limited resources.

We begin by letting x_1 and x_2 denote the amounts of the regular and super gasoline, respectively, that must be determined. This yields a profit l (in dollars) of

$$l = x_1 + 4x_2.$$

To make it more clear, we fill the data of the problem and the unknown amounts x_1 and x_2 in Table 1. Since only 24 units of resource R_1 are available and both types of gasoline require 6 and 4 units of R_1 for each unit, respectively,

$$6x_1 + 4x_2 \leq 24.$$

Similarly, by considering the amount of available resources R_2 and R_3,

$$5x_2 \leq 10 \quad \text{and} \quad 6x_1 + 10x_2 \leq 30.$$

Finally, since neither x_1 or x_2 should be negative, we add the requirements $x_1 \geq 0$ and $x_2 \geq 0$. The mathematical model of Product Mix Problem is given on the right side of Table 1.

	Regular	Super	Available
R_1	6	4	24
R_2	0	5	10
R_3	6	10	30
Profit	\$1	\$4	
Amount	x_1	x_2	

Table 1 Product Mix Problem

Mathematical model:

objective : Maximize $l(x) = x_1 + 4x_2$ subject to

constraints:

$$\begin{cases} 6x_1 + 4x_2 \leq 24 \\ \quad\quad\; 5x_2 \leq 10 \\ 6x_1 + 10x_2 \leq 30 \\ x_1 \geq 0,\ x_2 \geq 0 \end{cases}$$

The Diet Problem. This example considers a problem belonging to a class of *diet problems*. We have to determine the most economical diet in terms of calories that satisfies certain daily requirements for some nutrients such as proteins, calcium, iron, vitamins, etc. Imagine that today you can eat only four foods: trout, a corned beef sandwich, a ham and eggs, and a hamburger. The problem is what amounts of each food to eat so as to minimize your caloric consumption, keeping at least the minimum daily requirement of vitamins A, C and D.

Table 2 lists the amounts of each vitamin provided by a unit of each food that you can prepare, the minimum daily units of the vitamins A, C, D required to consume, and the calories in a unit of each food. The variables x_1, x_2, x_3, x_4 in Table 2 are the amount of each food, respectively, that you can eat in the optimal diet.

	Trout	Sandwich	Ham& eggs	Hamburger	Requirements
Vit.A	203	90	270	500	2000
Vit.C	92	84	80	90	300
Vit.D	100	230	512	210	430
Calories	600	350	250	500	
Amount	x_1	x_2	x_3	x_4	

Table 2 A Diet Problem

Mathematical model: Minimize $l(x) = 600x_1 + 350x_2 + 250x_3 + 500x_4$

subject to

$$\begin{cases} 203x_1 + \ \ 90x_2 + 270x_3 + 500x_4 \geq 2000 \\ \ \ 92x_1 + \ \ 84x_2 + \ \ 80x_3 + \ \ 90x_4 \geq 300 \\ 100x_1 + 230x_2 + 512x_3 + 210x_4 \geq 430 \\ \qquad x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0 \end{cases}$$

What is the common among the mathematical models considered above? Firstly, there is a linear function, that represents the goal (maximal profit or minimal calories) of the decision maker. Secondly, there are linear inequalities that represent the constraints imposed on reaching this goal.

In general, the LP problems are problems of maximizing or minimizing a **linear function** subject to a finite number of **linear equality/inequality constraints**. The function to be maximized (minimized) is called **objective function**.

The constraints of the type $x_j \geq 0$ are called ***nonnegative constraints*** and the remaining constraints are called ***main constraints***.

If a LP problem depends on n variables x_1, x_2,..., x_n, then each n-tuple of numbers $x = (x_1, \ldots, x_n)$ that satisfies all constraints is known as a **feasible solution.** For instance, the Diet Problem is a LP problem that depends on four variables x_1, x_2, x_3, x_4 and each quadruple of numbers $x = (x_1, x_2, x_3, x_4)$ that satisfy all constraints is a feasible solution of this Diet Problem. A feasible solution that gives the best objective function value is called an textbfoptimal solution.

The set of all feasible solutions of an LP problem forms its **feasible region**. If the constraints are incompatible, the feasible region is the empty set and the LP problem is called **infeasible**.

3 The Graphical method

In this section we present a graphical approach to solving the above LP problems based on a geometrical representation of the feasible region and the objective function.

3.1 Sketching the feasible region

Consider first the ***solution set*** of the linear inequalities

$$a_1x_1 + a_2x_2 \leq b \quad \text{and} \quad a_1x_1 + a_2x_2 \geq b.$$

The solution set of each one of them is a closed ***half-plane*** defined by the straight line

$$p : a_1x_1 + a_2x_2 = b.$$

To sketch the solution set of a linear inequality we do the following: (1) Sketch the straight line obtained by replacing the inequality with an equality; (2) Choose a test point which is not on the line. (3) Check if the test point satisfies the inequality. If this is true then the solution set is the half-plane on the same side of the line as the test point. Otherwise, the solution set is the half-plane on the other side of the line. (4) Shade the half-plane witch is not the solution set.

Note that a good choice for a test point would be the origin $O(0,0)$ if the line does not pass through it; otherwise, you could choose a point on one of the axes.

For example, let us determine the solution set of the inequality

$$x_1 - 2x_2 \leq 2.$$

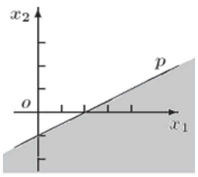

Fig.1 Solution set

(1) We sketch the line $p : x_1 - 2x_2 = 2$ (Fig.1) passing through the points $(2,0)$ and $(0,-1)$.

(2) We choose the test point to be $O(0,0)$.

(3) We substitute $x_1 = 0$ and $x_2 = 0$ in the inequality and get the true inequality $0 - 2 \times 0 \leq 2$, i.e. the solution set is the half-plane that contains $O(0,0)$.

(4) We shade the half-plane which does not contain the point $O(0,0)$.

Generally, to sketch the feasible region determined by a system of linear inequalities we do the following: we determine the solution set (the half-plane) represented by each inequality on the same graph, remembering to shade the half-plane that does not consist of points satisfying the inequality. The feasible region is the part which remains unshaded.

Example 1 The feasible region of the Product Mix Problem . It is marked here by Ω:

$$\Omega : \begin{cases} 6x_1 + 4x_2 \leq 24 & \Longrightarrow \quad p_1 : 6x_1 + 4x_2 = 24 \\ \qquad\quad 5x_2 \leq 10 & \Longrightarrow \quad p_2 : \qquad\; + 5x_2 = 10 \\ 6x_1 + 10x_2 \leq 30 & \Longrightarrow \quad p_3 : 6x_1 + 10x_2 = 30 \\ x_1 \geq 0,\ x_2 \geq 0 & \end{cases}$$

The solution sets of the inequalities are determined by the corresponding straight lines p_1, p_2 and p_3. The straight line p_1 passes through the points (4,0) and (0,6), p_2 passes through the point (0,2) and it is parallel to the axis Ox_1, p_3 passes through the points $(5,0)$ and $(0,3)$. The both inequalities $x_1 \geq 0$ and $x_2 \geq 0$ determine the 1-st quadrant of the coordinate system Ox_1x_2. The feasible region Ω is the unshaded pentagon *OABCD* (Fig. 2).

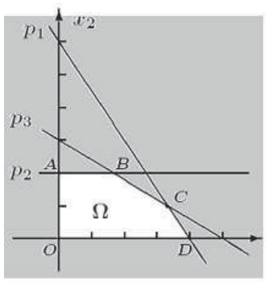

Fig. 2 The feasible region

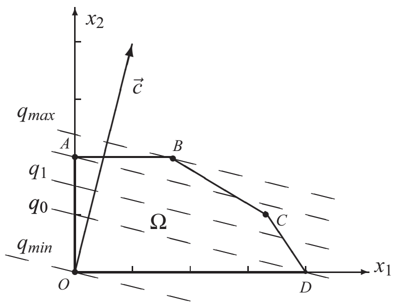

Fig. 3 Level lines of $l(x)$

The intersection of finitely many half-planes (half-spaces for the n-variables case) is called ***polygon***. The vertices of a polygon are also known as ***extreme points***.

3.2 Sketching the objective function

Example 2 The graphical solution of the Product Mix Problem. It is illustrated in Fig. 3. The objective function $l(x) = x_1 + 4x_2$ is maximized over the same feasible region Ω already constructed in Fig. 2. To make it more clear, the scale is larger and the redundant parts of the lines p_1, p_2 and p_3 are omitted.

Let us set the profit to be $l(x) = 4$, i.e. $x_1 + 4x_2 = 4$. In this way, we obtain a strait line $q_0 : x_1 + 4x_2 = 4$ passing through the points $(4,0)$ and $(0,1)$ and perpendicular to the vector $\vec{c} = (1,4)$ called ***normal vector*** of q_0 (see Fig.3). Note that any point (x_1, x_2) on q_0 represents a combination of the regular and super gasoline that yields always the same profit of \$ 4. Such a line is called a ***level line*** of the objective function $l(x)$.

Let us now set the profit $l(x)$ to be 6. We obtain the line $q_1 : x_1 + 4x_2 = 6$ passing through the points $(6,0)$ and $(0,1.5)$ and perpendicular to the same vector $\vec{c} = (1,4)$. Note that any combination of regular and super gasoline corresponding to a point on q_1 provides a profit of \$ 6. Thus, q_1 is another level line of the function $l(x)$.

We can obtain multimple level lines of $l(x)$. Several of them are drawn as dotted lines in Fig.3.

What is the common among the level lines of an objective function?

- All level lines have one and the same normal vector $\vec{c}$, whose coordinates are the coefficients of the objective function, therefore the level lines are parallel to each other. In our case $\vec{c} = (1,4)$.
- Moving from level line to level line in the direction of the normal vector $\vec{c}$, the values of the objective function increase. In our case, since q_1 lays on the right side of q_0, i.e. in the direction of $\vec{c}$, the value (\$ 6) of $l(x)$ at the points on q_1 is greater than its value (\$ 4) at the points on q_0.
- Vice versa, moving from level line to level line in the opposite direction of vector $\vec{c}$, the values of the objective function decrease.

Several level lines (dotted line) are drawn in Fig.3. Moving from line to line in the direction of $\vec{c} = (1,4)$, the values of $l(x)$ increase. The last available dotted line that has common points with Ω is named q_{max}. The only point of tangency between Ω and q_{max} is the extreme point B. Hence, B is the only optimal solution. Since B is the point of intersection of the lines p_2 and p_3 (see Fig. 2), we can solve the system of their equations and find its coordinates:

$$\left.\begin{aligned} 5x_2 &= 10 \\ 6x_1 + 10x_2 &= 30 \end{aligned}\right\} \implies x_1 = \frac{5}{3},\ x_2 = 2, \quad \text{i.e.} \quad B = \left(\frac{5}{3}, 2\right).$$

Substituting the coordinates of B into the objective function $l(x)$ we determine its maximal value: $l_{max} = \frac{5}{3} + 4 \times 2 = \frac{29}{3} \approx 9.7$. If the company produces $\frac{5}{3} \approx 1.67$ units of regular gasoline and 2 units of super gasoline, it will reach the maximal profit of \$ 9.7.

Example 3 The infinitely increasing profit. When a mathematical model of a real-life problem is created, it is possible to omit an important detail thus allowing for an infinite profit to be obtained. In this example, we consider such a mathematical model:

Maximize $r(x) = 4x_1 + 2x_2$

subject to

$$\Psi : \begin{cases} 2x_1 + x_2 \geq 2 \\ -x_1 + x_2 \leq 1 \\ x_1,\ x_2 \geq 0 \end{cases}$$

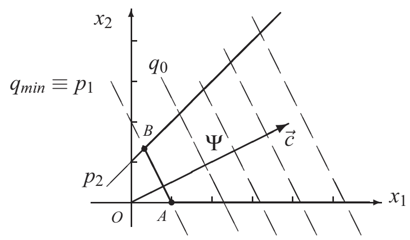

Fig. 4 Level lines of $r(x)$

The feasible region Ψ and the solution are given in Fig. 4. The feasible region Ψ has two vertices $A(1,0)$ and $B = \left(\frac{1}{3}, \frac{4}{3}\right)$, a finite edge AB and two infinite edges (rays) beginning from the vertices A and B, respectively. The level lines will always share some points with Ψin the direction of the normal vector $\vec{c}= (4,2)$. Therefore, $r(x)$ could take arbitrary large values over Ψ, i.e. the profit could increase infinitely $(r(x) \to +\infty)$.

Such unrealistic models are due to the omission of some essential constraints.

The algorithm behind the Graphical method

1. Sketch the feasible region. If it is empty, stop. (The LP problem is infeasible.)
2. Sketch the normal vector $\vec{c} = (c_1, c_2)$ to the level lines of the objective function $c_1x_1 + c_2x_2$.
3. Sketch an arbitrary level line q_0 that is perpendicular to $\vec{c}$ and crosses the feasible region.
4. Sketch several dotted lines parallel to q_0 in the direction of $\vec{c}$, if you maximize, and in the opposite direction, if you minimize. Two cases are possible:
   - There exists a dotted line tangent to the feasible region. All tangent points are optimal solutions. At least one of them is an extreme point (a vertex) of the feasible region. Determine its coordinates and substitute them into the objective function to determine its optimal value. Stop.
   - There is no such a dotted line. Stop. (The LP problem has no optimal solutions, because its objective function increases/decreases infinitely over the feasible region when you maximize/minimize. Such an LP problem is often called **unbounded**).

Such unrealistic models are due to the omission of some essential constraints.

Characteristics

Summarizing the above discussion, we have shown that a 2-variable LP problem could:

- have a unique optimal solution corresponding to an extreme point (a vertex) of the feasible region;
- have many optimal solutions corresponding to an entire edge of the feasible region and, at least one of them is an extreme point (a vertex) of the feasible region;
- be unbounded, i.e. if you maximize/minimize, the objective function increases/decreases infinitely over the feasible region;
- be infeasible (inconsistent) because of an empty feasible region.

4 For more ambitious mathematicians: n-variable LP problems

In 1939 a Russian mathematician, Leonid Kantorovich formulated a real-life problems leading to multivariable LP problems. Naturally, most LP problems are multivariable but cannot be solved by the graphical method. For instance, such a problem is the Diet problem of four variables (Section 2).

Fortunately, the n-variable LP problems exhibit the same important characteristics as the 2-variable ones. Based on this the American mathematician *George Dantzig* devised in 1947 the so called **simplex method** for solving n-variable LP problems.

The idea of the simplex method is quite simple. Let us discuss a n-variable LP problem that maximizes (minimizes) the objective function $l(x)$ over the feasible region Ω. The simplex method considers only the extreme points (vertices) of Ω. Firstly, an extreme point X^0 is found. Then the method proceeds from an extreme point to an adjacent extreme point along the edge of Ω that is *uphill*/*downhill* with respect to the objective function $l(x)$ generating a sequence of extreme points with increasing/decreasing objective values. Thus, in a finite number of steps, an optimal extreme point will be reached, or an edge will be chosen which goes off to infinity and along which $l(x)$ goes to $+\infty$ $(-\infty)$.