In search of shortest paths

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1 Introduction

As seen in the previous theme, problems on maxima and minima arise naturally not only in science and engineering and their applications, but also in daily life. A great variety of these have geometric nature and this theme focuses on problems of the following type: *Find the shortest system of paths connecting given points in the plane*. Such problems appear in designing real-world structures, such as routing of heating and plumbing pipes inside a building, tracing the layout between logic gates in circuits to minimize propagation time, determining the pathway for oil or natural gas pipelines that are as short as possible while considering the terrain they cross or avoid, and other *minimal networks*.

Here we present some elementary mathematical methods for solving shortest-path problems to encourage you to attack more challenging situations of this kind.

2 Shortest paths between two and three points

In this section we consider some shortest-path problems that can be solved by using the general principle that *the shortest path between two points in the plane is the line segment joining them*.The simplest form of this principle is the *triangle inequality* which says that the sum of two sides of a triangle is greater than the third one. More generally, the length of any broken line joining two points A and B is greater than the length of the line segment AB (Figure 1).

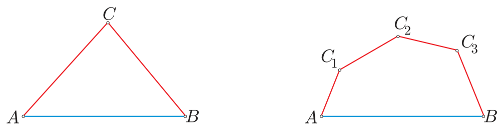

Figure 1

Let us start with the following practical problem.

Problem 2.1. *Design the shortest road between two "circular" lakes.*

Solution. Denote by L_1 and L_2 the circles representing the lakes and let O_1 and O_2 be their centers (Figure 2). Our goal is to find points M on L_1 and N on L_2 such that the length of the segment MN is minimal. Note

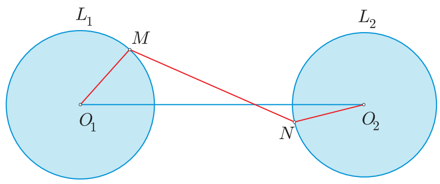

Figure 2

that this is equivalent to minimizing the length of the broken line O_1MNO_2 (Why?). But we know that its length is not less than the length of the segment O_1O_2. Hence the shortest road between the two lakes is the line segment M_0N_0, where M_0 and N_0 are the intersection points of the segment O_1O_2 with L_1 and L_2 (Figure 3).

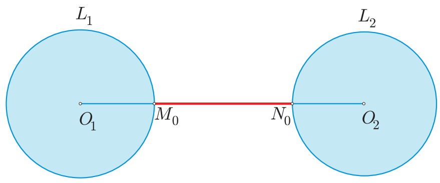

Figure 3

Next we consider some problems whose solutions involve geometric transformations such as reflection in a line or rotation about a point. The main idea is to "transform" the given problem into one of finding the shortest broken line connecting two points. A typical practical problem of the kind is the following.

Problem 2.2. *Two cities A and B are on one side of a highway. A company has to build a gas station G on the highway and a road from A to B trough G that goes straight from A to G and from G to B. Design such a road of shortest length.*

Solution. As usual in mathematics the cities A, B and the gas station G are considered as points, and the highway – as a straight line l (Figure 4). Hence the problem is to determine the position of the point G on l so that the sum $AG + GB$ is minimal. Let B' be the reflection of B in l.

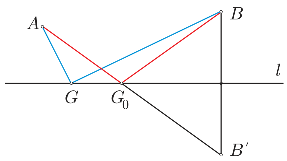

Figure 4

Then $BG = B'G$ and the triangle inequality for $\triangle AGB'$ implies that

$$AG + GB = AG + GB' \geq AB'.$$

Equality occurs precisely when G is the intersection point G_0 of l and the line segment AB'. Hence the company has to build the gas station at the point G_0.

The above problem has been considered about 2000 years ago by Heron [1] who stated that the shortest distance between A and B via l is exactly the path traversed by a ray of light emitted from A and observed at B. From here he deduced that when a ray of light is reflected in a mirror the angle of incidence is equal to the angle of reflection.

The next problem, known as *Pompeiu's theorem* [1], is essential for finding the shortest path between 3 points.

Problem 2.3. *For any equilateral triangle ABC and any point P in the plane the inequality* $AP + BP \geq CP$ *holds true. The equality is attained if and only if the point P lies on the arc* $\widehat{AB}$ *of the circumcircle of* $\triangle ABC$*, which does not contain C.*

Hint. Consider the 60° counterclockwise rotation about the vertex A , and let it carry P to P'. Then the given inequality is equivalent to the triangle inequality $PP' + P'C \geq PC$.

Let us consider now the following practical problem: *Given three cities* A, B, C, *design the shortest road system connecting them*. The first guess coming to mind is that the desired system consists of the two shorter sides of $\triangle ABC$ (Figure 5). However, we shall see that the answer is more complicated.

Consider an arbitrary system of roads (as curly as you want) connecting A, B, C. Then one can go from A to C and from B to C using some roads of the system (Figure 6).

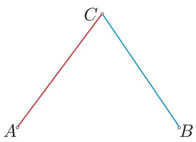

Figure 5

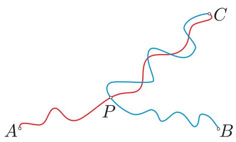

Figure 6

It is intuitively clear that we may assume for these two paths to lie entirely inside or on the boundary of $\triangle ABC$, otherwise we could find a shorter system of roads connecting the cities. Hence they intersect at C and possibly at some other points. Denote by P the one that is closest to A when going from A to C (Figure 6). Then the system of line segments PA, PB, PC is shorter than the given system and thus we come to the classical *Fermat problem*:

Given three points A, B, C in the plane find a point P such that the sum of distances from P to A, B, C is minimal.

The solution of this problem is well-known [1, 3] and it shows that the shortest road system connecting three cities A, B, C, consists either of the line segments PA, PB, PC, where P is the Toricelli point of $\triangle ABC$ (Figure 7), or of the two shortest sides of $\triangle ABC$ (Figure 8) depending on whether the largest angle of $\triangle ABC$ is less than 120° or not.

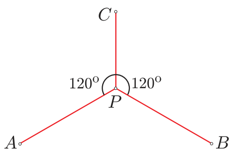

Figure 7

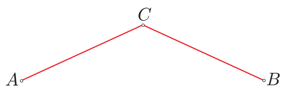

Figure 8

3 The Steiner tree problem

Having in mind the discussion in the previous section it is tempting to ask the general question of how one can design the shortest road system connecting an arbitrary number of cities. The same reasoning as above shows that to solve this problem it is enough to consider only road systems consisting of line segments such that any two cities are connected by a unique path. In mathematics, more precisely – in graph theory, such systems of line segments are called *trees*. Thus our question is reduced to the following problem posed by Jarnik and Kössler [7] in 1934 and known nowadays as the *Steiner tree problem*:

Given a finite set of points in the plane find the shortest tree spanning these points while allowing for addition of auxiliary points.

Each point added to construct the shortest tree is called a *Steiner point*. As we know, in the case of three points A, B, C such that the largest angle of $\triangle ABC$ is less than 120° we have to add one Steiner point, namely the Toricelli point of $\triangle ABC$. Next we show that if we have four cities A_1, A_2, A_3, A_4 which are at the corners of an imaginary square, then the shortest road system is a tree with vertices at these points and two Steiner points.

To prove this, consider an arbitrary system of roads connecting A_1, A_2, A_3, A_4. We may assume that the roads going from A_1 to A_3 and from A_2 to A_4 lie inside or on the boundary of the square, so they intersect at some points. Denote by P and Q the first and the last of these points when going from A_1 to A_3 (Figure 9). Then it is clear that the tree formed by the segments A_1P, A_4P, PQ, A_2Q, A_3Q (Figure 10) gives a shorter system than the initial one.

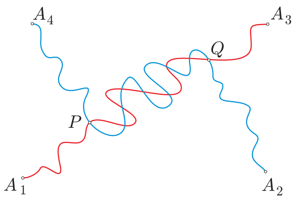

Figure 9

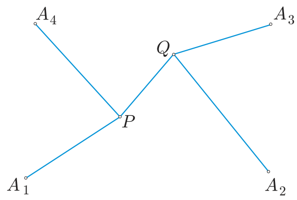

Figure 10

Hence, we have to find the shortest tree of this type. To do so, we shall use Pompeiu's theorem (Problem 2.3). Construct equilateral triangles A_1A_4A and A_2A_3B outside the square (Figure 11). Then

$$PA_1 + PA_4 \geq PA, \quad QA_2 + QA_3 \geq QB$$

and therefore

$$PA_1 + PA_4 + PQ + QA_2 + QA_3 \geq AP + PQ + QB \geq AB.$$

These inequalities show that the shortest tree is obtained when P and Q are the intersection points of the line segment AB and the circumcircles of $\triangle A_1A_4A$ and $\triangle A_2A_3A$, which are different from A and B (Figure 11).

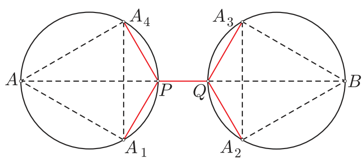

Figure 11

Thus in the case of a square we have two shortest trees spanning its vertices (Figure 12), each having two Steiner points. Note also that the angles between the segments meeting at a Steiner point are equal to 120°.

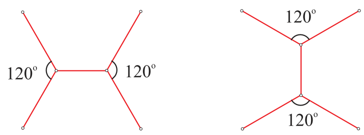

Figure 12

The above observations on the geometric structure of the Steiner minimal trees for 3 and 4 points (Figure 7 and Figure 12) hold true also for arbitrary number of points. More precisely, any Steiner minimal tree has the following properties.

• All of the original points are connected to 1, 2 or 3 other points.

• All Steiner points are connected to 3 other points.

• Any two edges meet at an angle of at least 120°. Hence the edges at a Steiner point meet at an angle of exactly 120°.

• The Steiner minimal trees of n points have at most $n-2$ Steiner points.

Finding the Steiner minimal tree in the general case is a complex problem since the given points may be placed anywhere in the plane. It was solved first by Melzak [8] in 1961 and since then many exact algorithms for solving the Steiner tree problem have been created. However it has been shown by Garey, Graham and Johnson [4] that this problem is NP-hard, meaning that there are no algorithms solving it in polynomial time. For instance, the well-known *Quicksort* algorithm for sorting n items is a polynomial time algorithm since it performs at most An^2 operations for some constant A. As noted in [5] the current best exact algorithm is known as *GeoSteiner version 3.1* and was implemented by Warme, Winter, and Zachariasen [9] in 2001. It can handle problem instances with up to 2000 points.

4 Tasks and problems

• *Let M be the midpoint of the line segment AB. Show that $CM \leq \frac{1}{2}(CA+CB)$ for any point C. When does equality occur?*

Hint. Consider the symmetric point C' of C with respect to M and apply the triangle inequality for $\triangle CC'A$. The equality holds iff C lies on the line AB, outside the open line segment AB.

• *Find the points X on the boundary of a square such that the sum of distances from X to its vertices is minimal.*

Hint. Use Heron's problem. *Answer*. The midpoints of the sides of the square.

• *Two "circular" lakes L_1 and L_2 are on one side of a highway l. A company has to build a gas station G on l and roads from the shores of L_1 and L_2 going straight to G. Design such roads of minimal total length.*

Hint. Apply Problem 1.1 for the lake L_1 and the reflection of the lake L_2 in l.

• *Two cities A and B are separated by a river having parallel banks. Design the shortest road from A to B that goes over a bridge across the river perpendicular to its banks.*

Hint. Denote by l_1 and l_2 the banks of the river and by l the line that is equidistant from l_1 and l_2. Let A_1 be the reflection of A in l. Then the given problem is reduced to Heron's problem for the points B, A_1 and the line l_2.

• *Find the points X in the plane such that the sum of distances from X to the vertices of:*
(a) *a given convex quadrilateral;*
(b) *a given centrally symmetric polygon*
is minimal.

Hints. (a) Use the triangle inequality to show that the desired point is the intersection point of the diagonals of the quadrilateral. (b) Use reflection in the center of symmetry of the polygon and the first problem.

References

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Recommended further reading

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