Christmas Stars

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1 Introduction

In order to find motivating tasks to their students, maths teachers may live in an ever ongoing search for fascinating topics and activities. Designing and building star shaped polyhedrons (for instance Christmas stars) could be one activity. If students are mainly accustomed to traditional paper-and-pencil exercises, they will probably approach an activity like this with mistrust and a dismissive attitude. It will be a challenge for the teacher to open the eyes of students to the mathematical opportunities connected with this sort of activity. Perhaps some teachers will even have to try it out for themselves before they can accept it as a serious approach to mathematics.

2 Mathematical Prerequisites

At this place you may expect a list of mathematical concepts and skills. Examples will come later because first of all, you will need an open mind and a desire for exploring designs and structures. A student of mine once said (before trying it out) that not much mathematics would be needed to construct this kind of stars. Using a ruler, a pair of scissors and a plate at home at the kitchen table, she postulated, it would be possible to make stars.

Trying it out will reveal that some planning of the design will be beneficial. Sitting with glue all over, realizing that elements do not fit together, is a rather unsatisfactory situation, and it appears that geometrical considerations are profitable.

On the other hand you sometimes have to build preliminary models to handle the geometry of the subject in focus. Bill Wheatland, an architect employed by Jorgen Utzon's Drawing Office 1964-66, said the following, "Where you've got complex geometry - as you for instance have on the surface of a car - it is much better to make a model: then you can see it, you can measure it, you can touch it – you can understand what it's all about", in a programme about the building of the Opera House in Sydney (Danish Television DR2, May 2003).

In making your first attempts to build stars, you should not go through a lot of mathematics beforehand. Try it out with simple means first. Starting with elementary geometric construction skills, you will be able to build your first copy of a star. Then you will realize that the more mathematics you master, the more you will be in control of the situation. The more advanced mathematics you introduce into the process, the more sophisticated models you can handle. The use of vector algebra and analytical geometry will be of great advantage at later stages.

3 Materials

Cardboard, glue, a pair of scissors are essential, but also usual tools as compass, ruler, pencil, calculator and computer.

Fig. 1 After geometry comes an exciting moment: Do things fit together?

4 Christmas Stars – Exploring Approach

Introduce an object to your students – for instance the star at the front in the picture below. The challenge will at first be to copy the star. Next you can make a copy in scale different to the original one; after this you may proceed to making stars with different numbers of points – and different shapes (longer or thicker arms). Your creativity may take flight at this stage of the process. For instance you may want to make a star like the one in the background of the photo in Fig 1.

Fig.2 Original star bought in an arts-and-craft store - and an altered version in the background

Fig.3 And a line of still more versions

4.1 Tasks

[1] Find a star and make a 1:1 copy of it.

[2] Make stars similar to the one in [1] but at different scales (larger and/or smaller).

[3] Make stars with different a number of points.

5 Christmas Stars – Some instructions

As a teacher you will be tempted to help students too much at the introductory stages, depriving them the ownership to processes and products. On the other hand, when they get stuck, you will have to offer support and encouragement at a psychological as well as mathematical level. Metaphorically speaking, terms like "scaffolding" and "zone of proximal development" can be found in the literature.

Some detailed description of possible approaches may be needed to help initiate the process. Due to lack of space in this volume a detailed step by step manual will not be printed here. You can find more on the web page for the project.

One important point is to notice the role of elementary geometry. Furthermore, today the use of geometrical software should not be omitted – on the contrary, the use of software can make the job more efficient and joyful, since it enhances precision and possibilities for experiments with details of design.

The photos below give some hints concerning the transition between the geometric description and physical objects.

Fig. 4 Decide the number n of points for the star (here a five-pointed star is depicted). Then decide the radii r and R of the shown circles lying in the plane of symmetry. From this you can construct a plane star which will serve as base, stabilizing the two halves of the space star.

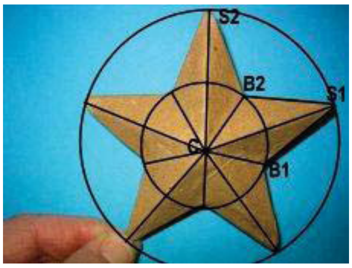

Fig. 5 The space star consists of two identical halves, each of which is in fact a pyramid with a star-shaped polygon as base. Nets for folding each of these halves should be constructed under geometrical considerations, cut out, folded and glued to the base star.

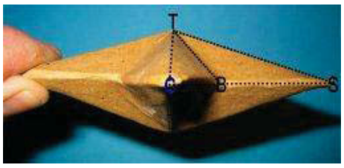

Fig. 6 The net for a half star consists of congruent triangles like TBS on the photo. Determining the sides of these triangles can be done by measuring or calculation. The calculation approach enables you to make precise adjustment in the design without a physical star at hand. Using dynamic geometry such as Geometers Sketchpad can be of great advantage.

Fig. 7 below shows a typical net of triangles for a star with five points like the one in the photo above. (Tabs for gluing are omitted).

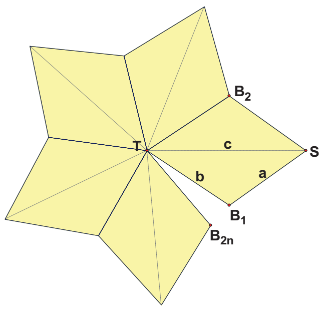

Fig. 7 Net of triangles for a five-star.

Relations between the different elements are summarized in the following figures, where C denotes the centre of the star. From this you can measure or calculate and construct.

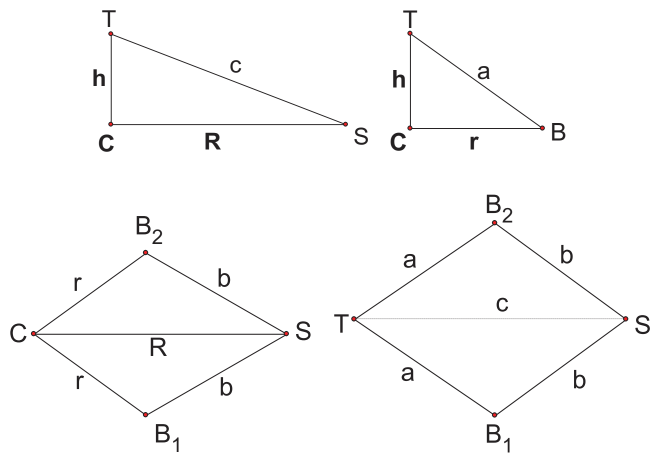

Fig. 8 Triangles essential for the construction of the net in Fig. 7. Also see Fig. 4 – Fig. 6.

Algebraic relationships can be established by using the Pythagorean Theorem and the Cosine Relation

$$(1)\quad a^2 = r^2 + h^2$$

$$(2)\quad c^2 = R^2 + h^2$$

$$(3)\quad b^2 = r^2 + R^2 - 2rR\cos\left(\frac{180°}{n}\right)$$

Fig. 9 Relationships derived from Fig. 8

5.1 Tasks

[1] Choose some values for r, R and h. Calculate a, b and c. Construct the star and check the size (measure r, R and h).
[2] Use a dynamic geometry programme to construct nets for folding.
[3] Choose some values for a, b and c. Can you determine values for h, r and R?

6 A deeper survey into a design problem: No slicing.

In this section I will briefly show that there are great opportunities for doing mathematical investigations on a level higher than primary school. Consider this problem: Could the net for a half star be constructed in such a way that there is no slice that needs gluing? The net in Fig. 7 above is with a slice. What are the conditions for avoiding a slice?

At first one sees (Fig. 7 and Fig. 8) that the net angle $\angle B_1TB_2$ must equal $\frac{360°}{n}$

Putting $\alpha = \cos\left(\frac{180°}{n}\right)$ and applying the cosine relation to triangle TB_1S we find that the following equation has to be fulfilled

(4) $$\alpha = \frac{a^2 + c^2 - b^2}{2ac}$$

where a, b and c are as in the Fig. 7 and 8

Substituting (1), (2) and (3) from Fig. 9 into (4) you get

(5) $$\alpha = \frac{r^2 + h^2 + R^2 + h^2 - \left(r^2 + R^2 - 2rR\alpha\right)}{2\sqrt{r^2+h^2}\sqrt{R^2+h^2}}$$

which by rearranging, squaring, expanding, etc. becomes

(6) $$\alpha^2 r^2 + \alpha^2 R^2 - 2rR\alpha = h^2\left(1-\alpha^2\right) = h^2\beta^2 \text{ with } \beta = \sin\left(\frac{180°}{n}\right)$$

Solving for h yields

(7) $$h = \sqrt{\frac{\alpha^2\left(r^2+R^2\right) - 2rR\alpha}{\beta^2}} = \sqrt{\frac{r^2+R^2}{\left(\frac{\beta}{\alpha}\right)^2} - \frac{2rR}{\frac{\beta}{\alpha}\beta}} = \sqrt{\frac{r^2+R^2}{\tan^2\left(\frac{180°}{n}\right)} - \frac{2rR}{\tan\left(\frac{180°}{n}\right)\sin\left(\frac{180°}{n}\right)}}$$

For the radicand in (7) to be non-negative conditions are

$$\alpha\left(r^2+R^2\right) > 2rR \Leftrightarrow \frac{r^2+R^2}{rR} > \frac{2}{\alpha} \Leftrightarrow \frac{r}{R} + \frac{R}{r} > \frac{2}{\alpha} \Leftrightarrow x + \frac{1}{x} > \frac{2}{\alpha} \Leftrightarrow x^2 - \frac{2}{\alpha}x + 1 > 0 \text{, with } x = \frac{r}{R}$$

The discriminant D of the quadratic polynomial in the last inequality is

$$D = \frac{4}{\alpha^2} - 4 = \frac{4\left(1-\alpha^2\right)}{\alpha^2} = \frac{4\beta^2}{\alpha^2} \text{ and the roots are } \frac{\frac{2}{\alpha} \pm \frac{2\beta}{\alpha}}{2} = \frac{1 \pm \beta}{\alpha}$$

From this you get the condition

$$\frac{r}{R} < \frac{1 - \sin\left(\frac{180°}{n}\right)}{\cos\left(\frac{180°}{n}\right)}$$

When this condition is fulfilled you can calculate h from r and R and construct a net without slice.

Another use of the formulas developed above could be, for fixed R (=1) graphing h as a function of r shoving that the smaller r is compared to R, the higher the star will be.

Recommendations for further reading

- Jenkins, G., Bear, M. Paper Polyhedra in Colour, Tarquin, UK,1999
- http://www.korthalsaltes.com/ (January 15, 2010)
- http://www.georgehart.com/ (January 15, 2010)
- http://www.software3d.com/ (January 15, 2010)