Navigation through numbers and fun

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1 Introduction

The main novelty in the approach to adapt the work of our math labs to real life problems and their math contents consists of the problem posing part in the first phase of their activities. More precisely, each team (math Lab) could propose one math problem and the teams were invited to prepare problems coming from the real life. Of special interest were the fields of navigation and astronomy.

One of typical difficulties arises from the fact that there was no requirement on the well-posedness of the problem or on the uniqueness of the solution. The flexibility of the rules in this phase allow to propose very beautiful problems or to treat very interesting situation without knowing a priori that unique solution exists at all.

Each of the teams-labs were formed by groups of 5-7 students and 1 teacher. This specific lab had to invent interesting situation from the real life and to pose a math problem associated with this situation.

Another group of 6-7 math profs from University formed the Commission of the competition and the main task of the Commission in this first phase was to modify the problem if the questions or descriptions were not very clear.

2 Curriculum items covered this unit

- Theory of numbers
- Analysis: properties of the sequences
- Combinatorics

3 Tasks and problems

We plan to show an example of a problem presented in a very attractive way were one of the math objects used in the problem: sequence of numbers was not defined in a precise way. Formally, the problem is not very well connected with the main argument – Navigation and Astronomy. But this fact shows the difficulty to propose math problems on a give subject. Somehow, the math lab proposing the problem started to prepare the problem as a typical number theory problem and succeeded to have a nice interpretation of the situation as a small fairy tale. We shall present initially the original variant (that needed improvement) and then we shall see why and the improvement was done.

3.1 Problem I: How to find the treasure?

The famous pirate Captain Secante was forced, after terrible tempest, to shore on one unexplored island. While his sailors were trying to repair the ship, he decided to see the island. Just in the centre of the atoll he found a big coffer (lock box), locked by a heavy padlock with unknown combination as well the following brass plate (see fig. 1 below). Below Captain Secante read the following numbers

5	0	123	22

The Captain was more famous (well known) with his pirate raids (attacks) than with his intelligence. So after reading with essential difficulties the text on the brass he decided to try immediately an occasional combination. In the last moment he was stopped by his Boatswain who said: “Wait, Captain! I can help you but only if we divide in equal parts the treasure.” On one of the sides of the coffer the Captain found the text: “This treasure contains golden coins. Their number is 2009!” The Captain without knowing the meaning of the exclamation mark immediately replied: “OK! We shall share the treasure, but since I am the Captain I shall take 1500 golden coins!” The Boatswain was more capable as a mathematician than as a corsair (pirate), so he agreed hiding his smile under the moustache and making some calculations on the sand he found formula for the terms $a_{i,}$ of the sequence. Then he easily found the last three digits of BIG number a_i with I being the number 2009! After he opened the treasure. Can you say which is the padlock combination and can you discuss the case when no information for the formula for the term $a_{i,}$ is given ?

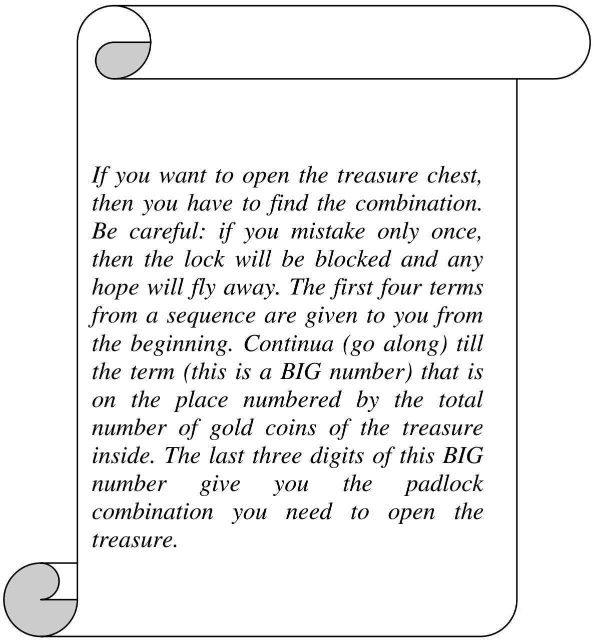

Fig. 1 see [1]

Formally, we see that this is not a situation from the real life. The math Lab (it was the team of a High School in Livorno, Italy) created beautiful story written very well so that a code for the treasure could be found. Somehow, one can imagine that first the math problem existed and then the story from the real (and not so real!) life was invented.

The problem of this proposal was the fact that it was not clear how to find rigorously the sequence. After some discussions the commission inserted the Italian version without any change, but put a modified English version of the problem, where the following precise definition of the sequence $a_{i,}$is given.

The Captain was more famous (well known) with his pirate raids (attacks) than with his intelligence. So after reading with essential difficulties the text on the brass he decided to try immediately an occasional combination. In the last moment he was stopped by his Boatswain who said: “Wait, Captain! I can help you but only if we divide in equal parts the treasure.” On one of the sides of the coffer the Captain found the text: “This treasure contains golden coins. Their number is 2009!” The Captain without knowing the meaning of the exclamation mark immediately replied: “OK! We shall share the treasure, but since I am the Captain I shall take 1500 golden coins!” The Boatswain was more capable as a mathematician than as a corsair (pirate), so he agreed hiding his smile under the moustache and making some calculations on the sand he found formula for the term $a_{i,}$, where $a_{i,}$ for even values of the index i is a sum of arithmetic progression with difference ± 1 and geometric progression of common ratio 5. Then he easily found the last three digits of BIG number a_i with i being the number 2009! After he opened the treasure. Can you say which is the padlock combination and can you discuss the case when no information for the formula for the term $a_{i,}$ is given ?

In order to help the understanding of the problem for some of teams (Bulgarian, Russian, Italian) the commission added some remarks and language references. For example some additional Internet links have been shown to illustrate the notions of arithmetic and geometric progression, padlock combination, metal plate called "brass plate", treasure chest, boatswain**.**

In some sense the above problem is closely connected with a problem posed in the preliminary talks of our meeting in Pisa, 2009. One of typical demands of teachers participating math labs was to give initial hints and examples of models and problems so that labs can use them as initial point in their work. The example was connected with the problem to find Island Treasure having a suitable math encoded describing the location of the treasure. A comparison of this initial example and the "Captain Secante" problem shows one possible way to provoke the creativity of pupils and teachers participating the lab.

4 Solutions

Here below we shall compare two of the given solutions (see [2]).

4.1 Solution to the problem by the Team Acutangoli (Livorno)

We can easily note that:

$$123 = 5^3 - 2,$$

$$22 = 5^2 - 3.$$

Following this kind of reasoning it is useful to write:

$$5 = 5^1 - 0,$$

$$0 = 5^0 - 1$$

and from this follows that the sequence is of type $5^{n+1} - n$ if n is even and $5^{n-1} - n$ if n is odd. Because the sequence begins with n=0 and on every step we add an unit to n, the 2009!th element of the sequence will be n=2009!-1. On the other side n is odd, because the factorial of every number greater than 1 is a multiple of 2 therefore the function f(n) that must be considered is:

$$f(n) = 5^{n-1} - n.$$

We know that 5^{α}=1000t+625, hence it finishes with 625, with α even and greater than 3 (see <u>Note</u>). We have that $\alpha = n - 1 = (2009! - 1) - 1 = 2009! - 2$ is even and greater than 3 hence $5^{n-1} = 1000m + 625$. Besides 2009! Is a multiple of 1000 hence 2009!-1 finishes certainly with the digits 999 and we can write it as: $1000n + 999$ We can write the 2009!th element of the sequence as:

$$5^{n-1} - n = 5^{(2009!-1)-1} - (2009! - 1) = 5^{2009!-2} - (2009! - 1) = 1000m + 625 - (1000n + 999)$$

from where:

$$1000(m-n) + 625 - 999 = 1000(m-n) - 374 = 1000(m-n-1) + 1000 - 374 = 1000(m-n-1) + 626$$

and therefore the last three digits of the 2009!th element are 626, because it is a positive number (indeed *m>n+1).*

Note. We can prove this by induction. Knowing that 5^4=625 we prove that if $5^{2\beta}$ finishes with 625 then also $5^{2(\beta+1)}$ will finish with 625, for β>1. From $5^{2\beta}$=1000t+625 follows that: $5^{2(\beta+1)}= 5^{2\beta+2}=5^{2\beta}\cdot 5^2$=(1000t+625)•25= =25000t+15625=(25000t+15000)+625.

4.2 Solution to the problem by the Team of Brescia

The requested sequence is the following $a_{n+1} = 5^{n+(-1)} - n, \forall n \in \mathbb{N}_0$. Replacing n with some concrete values, we calculate $a_1 = 5$, $a_2 = 0$, $a_3 = 123$, $a_4 = 22$. We know that a_{n+1} is defined by an algebraic sum of a geometric progression with ratio 5 and first term 5^{-1} and an arithmetic progression with common difference –1 and first term –n. The 2009!th element of the sequence is:

$$a_{2009!} = 5^{(2009!-1)+(-1)^{(2009!-1)}} - (2009!-1) = 5^{2009!-2} - 2009!+1,$$

and we want to find the last three digits

$$5^{2009!-2} - 2009!+1 \equiv 5^{2009!-2} + 1 \equiv 625+1 \equiv 626 \bmod (1000),$$

The combination is 626.

> A natural task is to see how the original problem could be solved or if it could be solved at all. This critical evaluation was done by the same team of Brescia in the second part of the solution.

If there is no information about the form of the sequence, there are different sequences with the following limitations:

$a_1 = 5$, $a_2 = 0$, $a_3 = 123$, $a_4 = 22$, $a_i \in \mathbb{N}$, such that there are different values for $a_{2009!} \bmod (1000)$ and therefore different combinations. For instance the sequences of the form: $a_{n+1} = 5^{n+(-1)^n} - n + f(n+1)$, where $f(n+1)$ is a function that satisfies

$$f(1) = f(2) = f(3) = f(4) = 0,\ f(n+1) \in \mathbb{N},\ f(2009!) \neq 0 \bmod (1000).$$

Now we will prove the existence of at least one function of that kind. The function f(x) is a sum of two quantities, one greater than or equal to 1, and the other equal to -1 hence $\forall x,\ f(x) \in \mathbb{N}$. For

$$x = 1, x = 2, x = 3, x = 4$$

we have $f(x) = 0$, hence the hypothesis is verified

$$f(1) = f(2) = f(3) = f(4) = 0,\ a_1 = 5,\ a_2 = 0,\ a_3 = 123,\ a_4 = 22.$$

Now remains to prove that there exists at least one α such that: $f(2009!) \neq 0 \bmod (1000)$. Suppose

$$\alpha^{(2009!-1)(2009!-2)(2009!-3)(2009!-4)} - 1 \equiv 0 \bmod (1000),$$

hence $\alpha^{(2009!-1)(2009!-2)(2009!-3)(2009!-4)} \equiv 1 \bmod (1000)$, which is impossible because the first term is certainly an even number raised to a natural exponent, and the second term is an odd number. This contradicts our assumption. It has been proven that different sequences respecting the hypothesis exist and therefore they give different final results and different combinations.

Note. This example shows that problem posing and problem solving labs had to interact and some of the critical remarks of the teams solving the proposed problems could essentially improve and modify the proposed problems.

5 Possible further tasks

Task 1. One can try to find the last four digits of the sequence?

Task 2. It is interesting to define in attractive, but rigorous way different types of sequences, as Fibonacci for example and treat a similar problem.

Task 3. One can ask participants of the course to invent other similar problems.

References

[1] http://www.dm.unipi.it/~eroe/index.php , site of the "Galilei" competition

[2] http://www.dm.unipi.it/~eroe/problemi.php , problems proposed by Math Labs