Spherical trigonometry: sum of angles of triangle on a sphere

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1 Introduction

Spherical trigonometry is closely connected with the astronomy. Today the study of astronomy requires a deep understanding of mathematics and physics. It is important to realise that Greek astronomy (in the period of 1000 years between 700 BC and 300 AD) did not involve physics. Indeed, a Greek astronomer aimed only to describe the heavens while a Greek physicist sought out physical truth. Mathematics provided the means of description, so astronomy during this period of 1000 years was one of the branches of mathematics. An essential development which was absolutely necessary for progress in astronomy took place in geometry. Spherical geometry was developed by a number of mathematicians with an important text being written by Autolycus in Athens around 330 BC. Some claim that Autolycus based his work on spherical geometry *On the Moving Sphere* on an earlier work by Eudoxus. Whether or not this is the case there is no doubt that Autolycus was strongly influenced by the views of Eudoxus on astronomy. The above introduction in a natural way turns our attention to this field and the problem posing labs formed by different teams participating the 2009 math edition of the "UnSungHero" proposed problems from this field.

2 Curriculum items covered this unit

- Spherical trigonometry
- Astronomy

3 Tasks and problems

Typical tasks in this domain are the following ones:

- Given latitudes and longitudes of two towns A and B find the distance between them.
- Solve spherical triangles.
- Find analogues of cos and sin theorems.

4 First step in spherical trigonometry: sum of angles of a triangle

4.1 Problem 1

Let A and B be points on the sphere with centre O and radius R. We shall call "segment AB on the sphere" the arc connecting A and B and lying on the plane AOB. If A,B and C are three points on the same sphere and AB and AC are "segments on the sphere", then the "angle" A between them is defined by the angle between tangent lines to the arcs AB and AC. Find on the sphere triangle ABC such that the sum of "angles" A, B and C is 270°.

Despite of the fact this is an easy problem, this is not a well studied argument in the High Schools. Nevertheless, some of the solutions have been very interesting and original. Let's consider one of them.

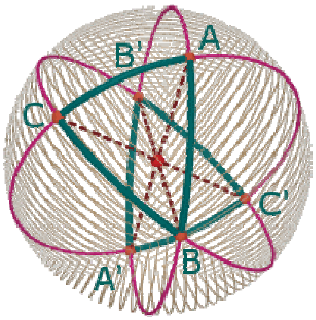
4.2 Solution of Problem 1 proposed by the team ACUTANGOLI (Livorno)

This is a problem of spherical geometry, thus the Euclidean theorem of the sum of the internal angles of a triangle does not hold: as a matter of fact, the sum of the internal angles of a spherical triangle is directly proportional to the area of the triangle.

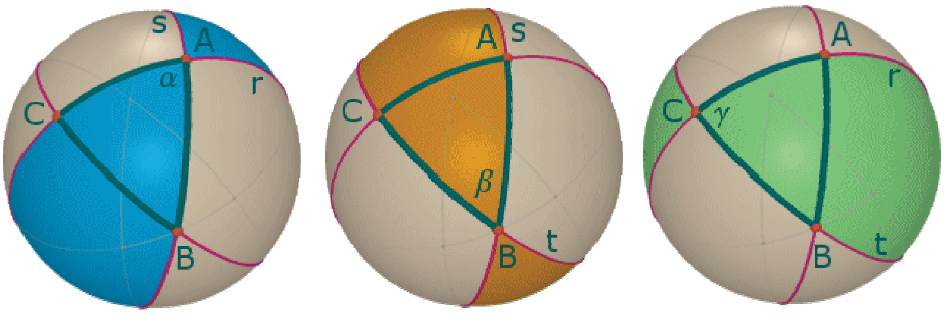
To simplify the calculations we consider R = 1. It is necessary to make it clear that with spherical wedge we have in mind: The portion of a sphere bounded by two maximal semicircles and with the poles of the wedge we have in mind the intersections between the two semicircles. By double spherical wedge we mean one wedge and his symmetrical wedge. The area of a spherical wedge S_F is directly proportional to the angle between the two semicircles, which can vary from 0 to π. We can set the proportion $S_F : S_s = \alpha : \pi$, with S_s area of the semicircle, in our case 2π. From where we obtain $S_F = 2\alpha$. For a double wedge we have

$$S_F = 4\alpha$$

Any spherical triangle can be obtained by the intersection of three double spherical wedges, each of them must have the pole in one different vertex, and opening corresponding to the angle of the vertex. Actually, the intersection of three double spherical wedges generates two spherical triangles which are inversely congruent, in the figure are shown with thick line.



We show the three double wedges from here above:



We notice from the figures, that the three double wedges cover perfectly the entire sphere, but the triangles *ABC* and *A'B'C'* are covered three times. The sum of the areas of the three wedges is equal to the surface area of the sphere, increased two times the area of *ABC* and two times the area of *A'B'C'*.Hence we can write $S = F_A + F_B + F_C - 2S_{ABC} - 2S_{A'B'C'}$. In our case $S = 4\pi$, $S_{ABC} = S_{A'B'C'} = S_T$, and because of the formula proved before $F_A = 4\alpha$, $F_B = 4\beta$ and $F_C = 4\gamma$. Substituting we obtain: $4\pi = 4\alpha + 4\beta + 4\gamma - 4S_T$ from where: $S_T = \alpha + \beta + \gamma - \pi$ It is required a triangle in which the sum of the internal angles is $270°$ or $\frac{3}{2}\pi$.

We substitute: $S_T = \frac{3}{2}\pi - \pi = \frac{\pi}{2}$. To have a sum of the internal angles of 270° it is necessary and sufficient that the surface area is $\frac{\pi}{2}$, or $\frac{1}{8}$ of the surface area of the sphere. One triangle that verifies that condition, which to simplify the explanation we will suppose drawn on the earth, is the one that has one vertex on the pole and the other two on the equator, on two perpendicular meridians.

4.3 Solution of Problem 1 proposed by the team Dini (Pisa, Italy)

Consider the spherical triangle ABC and denote by N the sum of the angles α,β,γ.

To ABC corresponds an antipodal triangle A'B'C' which is symmetrical to ABC with respect to the centre of the sphere. The two triangles have the same area Q. Let us consider the three double wedges which have as an angle between the two circumferences the angles α,β,γ. Knowing that the area S of the double wedge with angle θ**,** in the sphere with radius r' is $S= 4\,\theta\, r'^2$ we can express the total area of the sphere(**= 4πR²**) as the sum of three wedges, but we must remember to subtract 2 times the area of the spherical triangle ABC, because the wedges cover it two times, and two times the area of the spherical triangle A'B'C' for the same reason.

We obtain the following relations:

$$4\,\pi R^2 = 4\,\alpha R^2 + 4\beta R^2 + 4\,\gamma R^2 - 4Q$$

$$Q/R^2 + \pi = \alpha + \beta + \gamma = N$$

The triangles which have angles with sum N= 270°= 3/2 π rad, are those with area

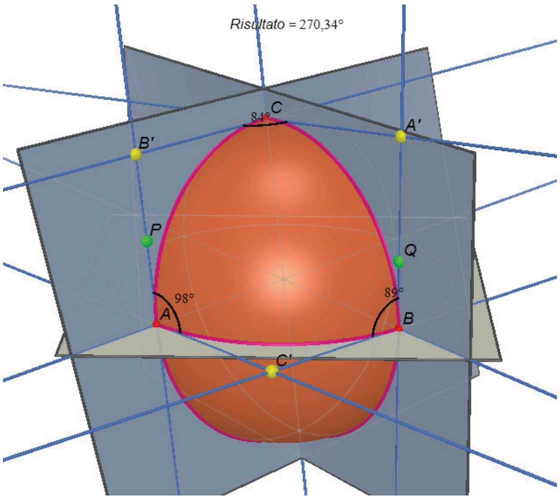
$$Q \rightarrow Q = (3/2\,\pi - \pi)\,R^2 = R^2\,\pi/2.$$

A more technical but interesting approach was proposed by another Italian team.

4.4 Solution of Problem 1 proposed by the team of Brescia, Italy

We can reformulate the problem as follows: "Find three points A, B, C on the spherical surface such that if we consider three planes passing through two of them and the centre of the sphere, the sum of the measures of the dihedral angles is $\frac{3}{2}\pi$. Using the Girard's theorem and imposing the condition that the sum of the dihedral angles must be the requested one we obtain: $A = Er^2$, $E = \alpha + \beta + \gamma - \pi$, where A is the area of the spherical triangle determined by the three points A, B, C, and r is the radius of the sphere and E is the spherical excess. Hence: $E = \frac{3}{2}\pi - \pi = \frac{1}{2}\pi$.We are looking for a triangle with area $\frac{1}{2}\pi$, and therefore $\frac{1}{8}$ of the total surface area of the sphere. Now it is clear that a triangle of that type is an equilateral one, with a side that measures $\frac{\pi r}{2}$, and one side on the equator and one vertex on the pole.

One example for research with Cabri:



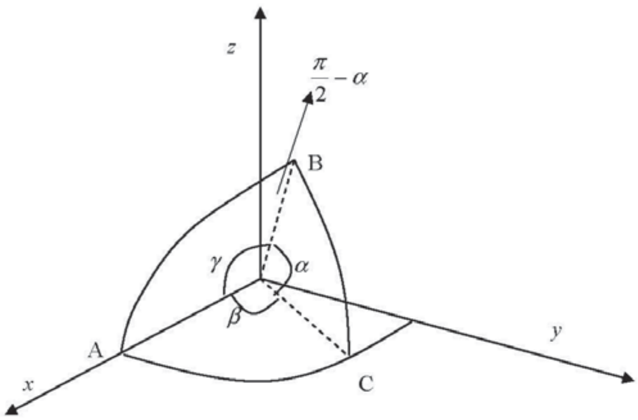
5 Next step: modified Pythagorean theorem on sphere.

5.1Problem 2

Let A and B be points on the sphere with centre O and radius R. We shall call "segment AB on the sphere" the arc connecting A and B and lying on the plane AOB. If A,B and C are three points on the same sphere and AB and AC are "segments on the sphere", then the "angle" A between them is defined by the angle between tangent lines to the arcs AB and AC. Given triangle ABC on the same sphere, such that the "angle" A is 90°, say if the Pythagorean theorem is true i.e. AB^2+AC^2 = BC^2, where AB, AC and BC are the lengths of the corresponding "segments"? If the theorem is not true, then how one can modify the Pythagorean theorem.

This argument and the problem to analyse the sum of angles of a triangle on a sphere are closely connected. The influence of the curved model of the notions and assertions of the elementary flat geometry attract the interest of High School students. Let's turn to the first solution of the problem.

5.2 First solution of Problem 2



We choose a rectangular triangle ΔABC with a right angle ACB and we shall call the length of its sides AC=b, BC=a and AB=c. We put it in a three-dimensional coordinate system centred at the centre of the sphere. So the coordinates of the point A are (R,0,0) and the point C lays in the plane determined by the axis x and y . We build the radii from the points C and B and let the central angle determined by the arc AB be γ , the central angle determined by the arc BC be α and the central angle determined by the arc AC be β . Each point can be determined by the distance between the point and the centre of the coordinate system and the angles between the axes x and z the projection of the point in the respective plane. So we can get the spherical coordinates of the points.

$$A(R,0,0)$$

We know the angles β and $\pi/2-\alpha$ which determine the point B so the spherical coordinates of B is:

$$B(R\cos\beta\sin(\frac{\pi}{2}-\alpha), R\sin\beta\sin(\frac{\pi}{2}-\alpha), R\cos(\frac{\pi}{2}-\alpha)) =$$
$$(R\cos\beta\cos\alpha, R\sin\beta, \cos\alpha, R\sin\alpha)$$

The angle that determines the point C is only β because it lays in the xy plane.

$C(R\cos\beta, R\sin\beta, 0)$

Using the theorem about the product of two points in a three-dimensional coordinate system we can find $\cos\gamma$

$$\cos\gamma = \frac{A.B}{\|A\|\|B\|} = \frac{R^2\cos\beta\cos\alpha}{R^2} = \cos\beta\cos\alpha$$

When expressing the angles in radians we will have $\alpha = \frac{a}{R}, \beta = \frac{b}{R}, \gamma = \frac{c}{R}$

So we will have the equation

$$\cos(\frac{c}{R}) = \cos(\frac{a}{R})\cos(\frac{b}{R})$$

This is called the spherical Pythagorean theorem. When expressing the $\cos\gamma, \cos\beta$ and $\cos\alpha$ by the Taylor series for the cosine function $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$ So we get:

$$1 - \frac{c^2}{2R^2} + \frac{c^4}{24R^4} - \ldots = (1 - \frac{a^2}{2R^2} + \frac{a^4}{24R^4} - \ldots)(1 - \frac{b^2}{2R^2} + \frac{b^4}{24R^4} - \ldots) = 1 - \frac{a^2}{2R^2} - \frac{b^2}{2R^2} + \frac{a^2b^2}{4R^4}$$

$$\Rightarrow c^2 + \frac{polynom\ of\ the\ high\ powers\ of\ c}{R^2} = a^2 + b^2 + \frac{polynom\ of\ the\ high\ powers\ of\ a\ and\ b}{R^2}$$

When a, b and c are constant and R gets larger compared to them, the surface of the sphere is getting closer to a plane and the value of $\frac{polynom\ of\ the\ high\ powers\ of\ c}{R^2}$ and $\frac{polynom\ of\ the\ high\ powers\ of\ a\ and\ b}{R^2}$ is getting closer 0. That is why we get the classical Pythagorean theorem as a limiting case of this theorem.

The solutions were made by the Bulgarian Team from Aprilov National High school, Gabrovo.

Analysis: The solution involves the use of three dimensional vectors and their scalar product. The Taylor expansion is also and argument used in University teaching in math. It is interesting to see how the team can try to solve a triangle with given two sides and angle between them on a sphere?

5.3 Solution of Problem 2 proposed by the team Dini, Pisa

Prove that Pythagorean theorem ($AB^2+AC^2=BC^2$) is not true for a sphere.

To do this is enough to find a counterexample: consider a triangle ABC on the sphere with radius R such that: every side is ¼ from the maximal circumference i.e. πR/2, the area is 1/8 from the total surface, i.e. $\pi R^2/2$. Note that all three angles are π/2 rad, and that the triangle is equilateral, therefore the sum of the three squares of the two sides is greater than the square of the third, as we wanted to prove.

We draw a Cartesian coordinate system with origin O which is also the center of the sphere, x axis passing trough O, such that C has coordinate z=0.

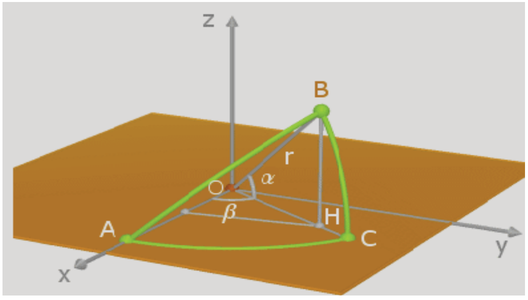
To find the coordinates of point B, we project it on the plane xOy, and we obtain the segment

$$OH = R\cos\alpha,$$

$$X_B = r\cos\alpha\cos\beta,$$

$$Y_B = r\cos\alpha\sin\beta,$$

$$Z_B = r\sin\alpha.$$



The scalar product is AB= |A| |B| cos γ , therefore we can express cos γ as function of the coordinates of A and B →

$$\cos\gamma = r^2\cos\alpha\cos\beta \,/\, \sqrt{(r^2\cos^2\alpha\cos^2\beta + r^2\cos^2\alpha\sin^2\beta + r^2\sin^2\alpha)} = r^2\cos\alpha\cos\beta / r^2 = \cos\alpha\cos\beta,$$

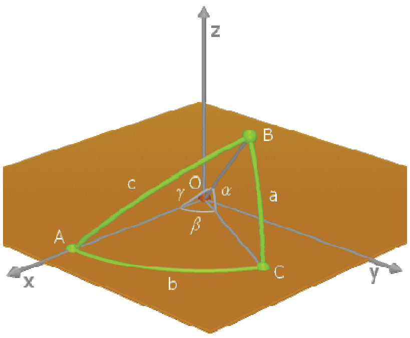
and because of the definition of radian → $\alpha = a/r$ $\beta = b/r$ $\gamma = c/r$ so we find the relationship

$$\cos(c/r) = \cos(a/r)\cos(b/r),$$

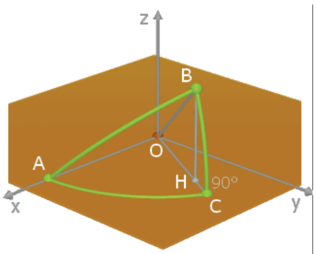
that is the relationship between the three sides of the triangle on the sphere.

5.4 Solution of Problem 2 by the team of Brescia, Italy

The standard formulation of the Pythagorean theorem is not true for spherical geometry. The existence of rectangular equilateral triangles shows that it is not always true. We can consider one rectangular equilateral triangle and indicate his sides a, b, c, we know that a=b=c, we have $a^2 + b^2 > c^2$.Therefore our goal is to find a relationship that binds the hypotenuse and the legs (or catheti) of any rectangular triangle on the sphere. Consider a Cartesian coordinate system xyz, a sphere with centre O in the origin, and one rectangular triangle ABC with his right angle in C. Using an appropriate isometry we can place the triangle ABC in such way that his vertex A is in the point (r, 0, 0) and the side AC lies on the plane xy as can be seen on the first figure.

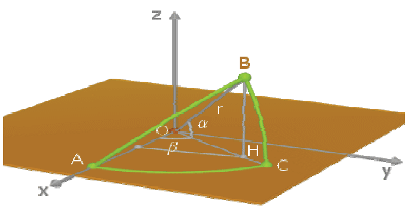


We denote with α, β, γ the angles in the centre formed by the sides (arcs) BC, AC and AB (see the third figure).The length of an arc of a circle subtending angle α, in the centre, can be obtained by the following proportion:



$s : 2\pi r = \alpha : 2\pi$, from where $s = \alpha r$. In our case we have:

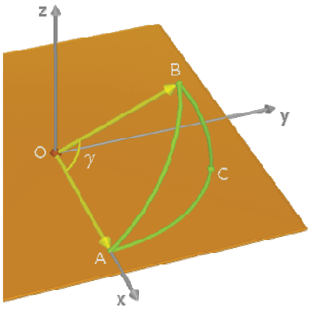
$a = \alpha r$
$b = \beta r$
$c = \gamma r$ (*)



Now we search the coordinates of the points A and B. The coordinates of A are given by construction. Now we will determinate the coordinates of B.

Let the orthogonal projection of point B over the plane xy intersect the plane xy in point H.

H lies on OC, therefore the angle BOH is equal to α (see the fourth figure).Because OH = r cos we deduce that the coordinates of B are $B = (r\cos\alpha\cos\beta; r\cos\alpha\sin\beta; r\sin\alpha)$.



Now we can find the angle γ. Consider vectors A and B, both with initial points in O and ending respectively in A and B, we have $AB = |A||B|\cos\alpha$ hence $\cos\alpha = \frac{AB}{|A||B|}$.

From $|A| = |B| = r$ and the scalar products of vectors we obtain

$$\cos\gamma = \frac{r^2\cos\alpha\cos\beta}{r^2},$$

$\cos\gamma = \cos\alpha\cos\beta$ and from (*) $\cos\left(\frac{c}{r}\right) = \cos\left(\frac{a}{r}\right) + \cos\left(\frac{b}{r}\right)$,

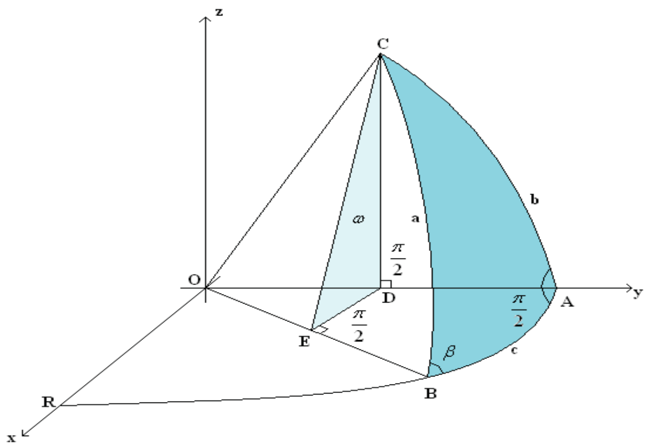
and this is the relation we were looking for.

Analysis: The inequality $a^2 + b^2 > c^2$ should be studied more carefully. It is not obvious for example, what is necessary and sufficient condition (imposed on the angles of the triangle) so that this inequality is true?

One can try to find more elementary argument. This is the way chosen by Italian team of Livorno.

5.5 Solution of the Problem 2 by the Team ACUTANGOLI (Livorno)

To verify that the Pythagorean theorem does not hold it suffices to find a counterexample: let us consider a triangle with tree right angles, obtained by the intersection of a sphere with tree perpendicular planes, the intersection of which corresponds to the centre of the sphere. Take the radius , it is clear that every side is $\frac{1}{4}$ of the maximal circumference, hence $l = \frac{1}{4}2\pi = \frac{\pi}{2}$. It is $AB = BC = CA = l$, and clearly the equation does not hold $AB^2 + AC^2 = BC^2$. Thus we have proved that the Pythagorean theorem is not true for a sphere, now we have to find out if it is possible to modify it in a way that it is true. Refer to the following figure:



Let the planes OAC and OBA be perpendicular, hence the spherical angle $C\hat{A}B$ is right, and the triangle ABC, coloured in dark blue, is a spherical right-angled triangle. Consider the plane ω passing through C and perpendicular to the line OB. This plane, coloured in light blue, is perpendicular to the plane OBA for the theorem "Two planes α and β are perpendicular if one of them contains one straight line orthogonal to the other". Let $D = \omega \cap OA$ and $E = \omega \cap OB$. To simplify the calculations, consider from now on the radius of the sphere to be 1.

We can assert that $a = BOC^{RAD}$; $b = COA^{RAD}$; $c = AOB^{RAD}$. At this point:

- triangle OCE, right-angled in E $\Rightarrow$ $OE = \cos a$,
- triangle OCD, right-angled in D $\Rightarrow$ $OD = \cos b$,
- triangle OED, right-angled in E $\Rightarrow$ $OE = OD \cdot \cos c$.

Substituting the first two in the third one we have:

$$\cos a = \cos b \cdot \cos c,$$

that is a relation that connects catheti and the hypotenuse of our triangle, very similar to the Pythagorean theorem. We can enunciate: "In one right sphere triangle, the cosine of the hypotenuse is equal to the product of the cosines of the catheti"

If we had not supposed $R = 1$, the "Pythagorean theorem on the sphere" would be:

$$\cos\frac{a}{R} = \cos\frac{b}{R} \cdot \cos\frac{c}{R}.$$

At this point we must add: arises the question where did the Pythagorean theorem go as it is taught in the middle school, because at the end, when we draw a right triangle on a sheet of paper it is a spherical one because the earth is a sphere, hence the table on which we are drawing is curved in some measure. Continuing on this way we notice that R is much bigger than the sides of the triangle, thus the fractions $\frac{a}{R}$, $\frac{b}{R}$ and $\frac{c}{R}$ are very small numbers. Considered that this fractions are very small we can hypothesize that we can obtain again the famous Pythagorean theorem. At this point we must say that, considered x very small, $\cos x \cong 1 - \frac{1}{2}x^2$ is a valid approximation. We rewrite the "Pythagorean theorem on the sphere", approximating the cosines as above, we obtain:

$$1 - \frac{\left(\frac{a}{R}\right)^2}{2} \cong \left(1 - \frac{\left(\frac{b}{R}\right)^2}{2}\right) \cdot \left(1 - \frac{\left(\frac{c}{R}\right)^2}{2}\right)$$

from where:

$$1 - \frac{\left(\frac{a}{R}\right)^2}{2} \cong 1 - \frac{\left(\frac{b}{R}\right)^2}{2} - \frac{\left(\frac{c}{R}\right)^2}{2} + \frac{\left(\frac{b}{R}\right)^2\left(\frac{c}{R}\right)^2}{4}$$

we eliminate the two 1, and ignore the term $\frac{\left(\frac{b}{R}\right)^2\left(\frac{c}{R}\right)^2}{4} = \frac{b^2c^2}{R^4}$ and because R ˙ a,b,c and multiplying by two we obtain:

$$-\left(\frac{a}{R}\right)^2 = -\left(\frac{b}{R}\right)^2 - \left(\frac{c}{R}\right)^2$$

from where easily can be found $a^2 = b^2 + c^2$.

6 Analysis and conclusions

- these arguments are not studied deeply in the High School, neither in the University.
- It is important to mention that the curvature of the sphere is constant and positive. This Universe is bounded and positively curved
- It is not clear how the situation will change if the curvature is negative? One can try to see possible examples
- Obvious relations with astronomy are manifested.

It is interesting to implement such arguments in a course for preparation of future teachers and see how concrete problems from astronomy can be implemented in their work.

7 Tasks and further applications

- Suppose a town X has longitude 2°W, latitude 50°N. while the town Y has longitude 97°W, latitude 50°N. How far apart are they, in nautical miles, along a great-circle arc?
- If you know the distance between X and Y , the longitude and latitude of X and only latitude of Y can you find the longitude of X?
- Can you find a surface different from the sphere so that the sum of angles of a curvilinear triangle is less that 180°?

References:

[1] http://en.wikipedia.org/wiki/Spherical_trigonometry (possible further reading)

[2] http://www.dm.unipi.it/~eroe/index.php , site of the "Galilei" competition

[3] http://www.dm.unipi.it/~eroe/problemi.php , problems proposed by Math Labs