Astronomy and how instruments draw quadratic curves.

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1 Introduction

The problem posing section of the team competition can be considered as a real lab for studying the impact of real life problems to their math contents. It is interesting to see how the solutions found after this phase by different math Labs could further provoke more and deeper studying of some new phenomena connected with the initial problem.

2 Curriculum items covered this unit

2.1 Mathematical content

- Theory of quadratic curves.
- Systems of quadratic equations.

2.2 Scientific content

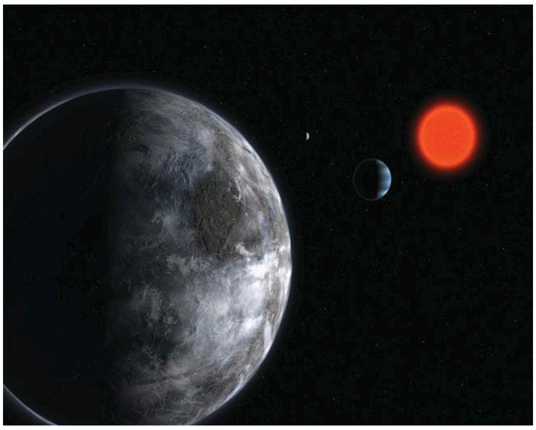
- Measurement theory

3 Tasks and problems

As a starting point let us consider the following problem proposed by the team of a small town Gabrovo in Bulgaria.

3.1 Problem 1

A student of the famous Galileo Galilei discovered a new planet, which orbits the Sun on an elliptic orbit with semi-axes *a* and *b*. Suppose that an observer is located at a point in the plane of the ellipse, such that the ellipse is seen from that point at an angle of $90°$. Compute the distance between the center of the ellipse and the observation point.



A standard solution was found by the team of Brescia, Italy

3.2 Solution to problem 1 (team pf Brescia, Italy)

We use Cartesian Coordinate System. The semi axes of the ellipse lie on the Cartesian axes, the centre of the ellipse is in the origin. The equation of the ellipse is:

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, let $P(x_0; y_0)$ be the point where the observer is, from where we draw the two tangent lines. Now we are looking for the angular coefficient of the tangent lines using the following system:

$$\begin{cases} \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \\ y-y_0=m(x-x_0) \end{cases}$$

where the tangency coefficient is:

$$m_{1/2}=\frac{-x_0y_0\pm\sqrt{x_0^{\,2}y_0^{\,2}-\left(a^2-x_0^{\,2}\right)\left(b^2-y_0^{\,2}\right)}}{a^2-x_0^{\,2}}.$$

The tangent lines from the point must be perpendicular so for their two angular coefficients is valid the relation: $m_1\times m_2=-1$.

From where

$$\frac{-x_0y_0+\sqrt{x_0^{\,2}y_0^{\,2}-\left(a^2-x_0^{\,2}\right)\left(b^2-y_0^{\,2}\right)}}{a^2-x_0^{\,2}}\times\frac{-x_0y_0-\sqrt{x_0^{\,2}y_0^{\,2}-\left(a^2-x_0^{\,2}\right)\left(b^2-y_0^{\,2}\right)}}{a^2-x_0^{\,2}}=-1,$$

so we get:

$$x_0^{\,2}+y_0^{\,2}=b^2+a^2.$$

The last equation is the locus of points in which the observer can be and for which is true that the tangent lines of the ellipse are perpendicular. This locus is the circle wit center in the origin and radius $r=\sqrt{a^2+b^2}$ .The distance between the observer and the center of the ellipse is the same as the distance between a point on the circle and the center hence this distance is the radius and is $\sqrt{a^2+b^2}$.

3.3 The solution proposed by the team of Livorno , Italy

The Lab proposing the solution used the notion of "circolo direttore dell'ellisse" (director circle of an ellipse).

We define *director circle* of an ellipse or a hyperbola as circle with radius equal to the main axis and centre one of the two foci. (see also [1])

The locus of points of a focus of an ellipse or a hyperbola symmetrical with respect to the tangent lines of the ellipse or hyperbola is the director circle which has as centre the other focus. An ellipse or a hyperbola is the locus of points that are equidistant from one of the two foci and from the corresponding director circle.

A director circle is a circle consisting of the points of intersection of pairs of perpendicular tangents to an ellipse or hyperbola.

The specific instrument to draw this circle is shown on the picture below.

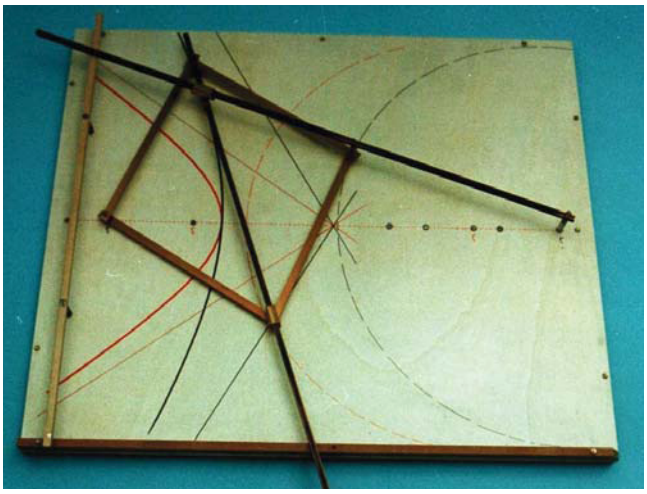
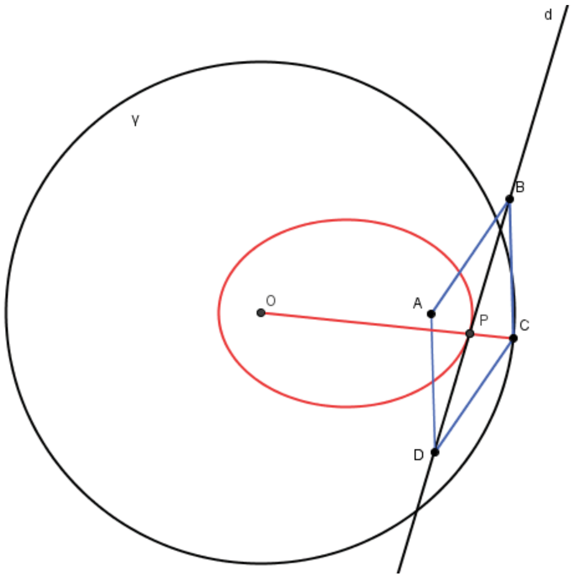


Fig. 1 (see [1] and [4])

The bar OC is fixed in O. The rhomb ABCD is hinged (the bars can move), the vertex A is fixed on the plane, the opposite vertex is bound to C by the bar OC with distance l from O (l<OA). The vertex D can be moved along the bar d which is bound to vertex B. When C traces the circumference γ with centre O and radius l, the point P, point of intersection of OC and BD traces a hyperbola with foci O and A and transverse axis l. BD is tangent in P to the conical shape and γ is her director circle.



One can find alternative information in Internet (see the references below).

4 Possible tasks

[1] What will happen if we replace the angle of $90°$ .in Problem 1 with angle of 60 degrees?

[2] Prepare a model of the instrument on figure 1.

[3] Try to study the case of hyperbola in the place of an ellipse more deeply.

References:

[1] **http://www.math.uoc.gr/~pamfilos/eGallery/problems/Director.html**

[2] **http://www.dm.unipi.it/~eroe/index.php** , site of the "Galilei" competition

[3] **http://www.dm.unipi.it/~eroe/problemi.php** , problems proposed by Math Labs

[4] **http://www.math.uoc.gr/~pamfilos/eGallery/problems/EllipseEvolute.html**