From fun problems to deep open problems

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1 Introduction

The main surprise of the problem posing lab in the team competition “Galilei” in 2009 was a problem that seemed to be innocent initially and prepared in manner different from the classical approach of the previous sections: the students and the teacher started from real situation and arrived at a deep math problem that have been intensively discussed by the colleagues of the University after the problem appeared. Let’s start with the math problem discussed by them: If you have a square on the plane, on can look at all curves γ, such that ε – neighbourhood of the curve covers the square. The problem is to find the curve having minimal length so that ε – neighbourhood of the curve covers the square.

The main surprise was the fact that this math problem seem to be completely open and only the case, when the parameter ε is very small was considered in the math research articles known by the colleagues of University of Pisa.

The example that we shall try to describe in this section is very important for problem posing labs and modelling in other braches of science and real life. Standard math curricula at Secondary School level has a very well organized structure based on lectures and exercises with concrete examples and problems with fixed answers. Math modelling in the problem posing labs is very risky process, since:

- important physical phenomena have very complicated models and need higher level tools of Mathematics,
- there are models having no rigorous math treatment and solution.

2 Curriculum items covered this unit

2.1 Mathematical content

- Geometry (on plane and in the space).

2.2 Scientific content

- Astronomy and navigation.
- Classical mechanics.

2.3 Tools to solve the problem

- Variational methods.
- Ordinary differential equations.

3 The problem and attempts to solve it

Let us now turn to the “innocent” problem proposed by the team of the High School “Dini” in Pisa.

3.1 Problem “Moon Satellite”

The satellite THETA of the society GoogleMoon is equipped with digital camera having image with a 20 degree angle of view. To make qualitative photos the satellite must circle around the Moon keeping constant altitude equal to the radius R=1738 km of the Moon and the digital camera must be oriented all the time in the direction of the centre of the Moon. Find the minimal length of the path of the satellite allowing to cover (with the images of the digital camera) the whole surface of the Moon

Fig.1 NASA picture of the Moon (see [3])

Remarks: If the cone of the camera has image with a 20 degree angle of view, then the cone of the camera is a cone with angle of semiaperture 10 degree.

Fig.2 NASA satellite (see [3])

One can see that this problem is similar to the problem known by researchers in Pisa University. The open problem treated the square on the plane, while the problem of the team of "Dini" looks for the case of a sphere. Therefore, the real life problem lead to deep real math problem!

The difficulty of the problem posed the question how the teams could solve the problem, since they have no deep math tools, but even the specialists with heavy math preparation could not solve it. The team of Brescia proposed an interesting simulation using orange.

3.2 Attempt to solve Problem "Moon Satellite" by the team of Brescia, Italy

Consider the intersection of the lunar sphere and the plane passing through the centre of the satellite point and consider the line that form the cone of the digital camera, The spherical cap of the moon which can be seen by the camera is obtained intersecting the two lines and the circumference. Solving the system we obtain the coordinates of the points of intersection and therefore the tangent of the semi angle of the cap which framed by the camera.

Now find the minimal path of the satellite! We think that it can be a spiral path but we can not calculate it. We tried to peel a lot of oranges. First we took off a cap of about 20° then we continued forming a spiral with a stripe with approximately the same dimensions. We measured the length of the orange peel that we obtained and we calculated the ratio between the orange radius and the radius of the sphere on which the satellite is orbiting but certainly this is not the right way.

3.2 Attempt to solve Problem "Moon Satellite" by the team of Dini, Pisa, Italy

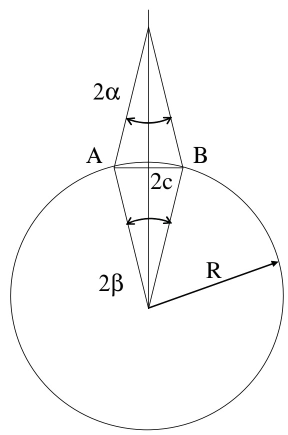
The team of "Dini" proposed an idea that is closer to the proposal that researchers at Math Department of Pisa University thought is the correct one.

Problem's data:

$R = 1738km$ Radius of the Moon

$2\alpha = 20°$

$H = R$ Hight of the satellite from ground



Trigonometrically is obtained the semi chord AB named c and the semi angle in the centre β

$$\sin\beta = \frac{c}{R}$$

$$c = R\left(2 - \sqrt{1-(\sin\beta)^2}\right)\tan\alpha$$

from where

$$\frac{c}{R} = \sin\beta = 0.179$$

Therefore the semi arc is:

$$a = R\beta = 313.1km$$

The area of the spherical cap framed by the camera is:

$$A_c = 2\pi R^2\left(1 - \sqrt{1-\left(\frac{c}{R}\right)^2}\right) = 3.072\cdot 10^5 km^2$$

From where is possible to obtain the minimal theoretical value L_{mt} of the projection of the satellite's trajectory on the Moon.

$$A_c + L_{mt}\cdot 2a = 4\pi R^2,$$

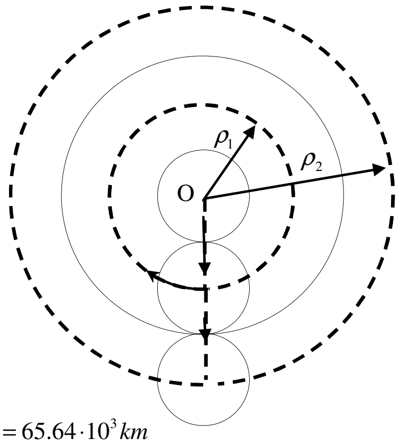
and hence

$$L_{mt} = 34.6R = 6.013\cdot 10^4 km.$$

We can not assure that such value can be obtained by a continuous movement of the satellite. One possible strategy of movement is to cover the lunar surface on parallel adjacent circumferences moving along the meridian and beginning from one pole.

We calculate the radius of every circumference and obtain:

i	ρ_i (km)
1	612.8
2	1.147 10^3
3	1.534 10^3
4	1.723 10^3
5	1.692 10^3
6	1.443 10^3
7	1.009$10^3$
8	445.4



From where we obtain that the path is:

$$L_1 = 2\pi \sum_{i=1}^{8} \rho_i + (\pi - \beta + \gamma) R = 37.8R = 65.64 \cdot 10^3 km$$

where is the last aperture of the circle around the opposite pole which is not covered after moving on the eight circle.

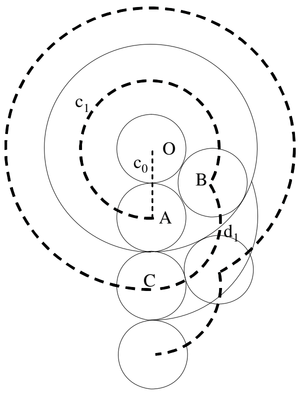
The efficiency of this strategy is given by:

$$e_1 = \frac{L_{mt}}{L_1} = 0.916$$

One possible increase of efficiency is given by the following strategy: the passage between the levels on each turn is made by circles tangent externally to the one from which we start

The values of the path are given in the following table:

i	c_i (km)	d_i (km)
0	612.8	
1	3.227 10^3	1.585 10^3
2	6.583 10^3	1.295 10^3
3	9.013 10^3	1.211 10^3
4	1.021 10^4	1.184 10^3
5	1.001 10^4	1.188 10^3
6	8.445 10^3	1.227 10^3
7	5.717 10^3	1.341 10^3
8	2.175 10^3	137.3



From where

$$L_2 = c_0 + \sum_{i=1}^{8} (c_i + d_i) = 37.5R = 65.17 \cdot 10^3 km$$

which gives us a slight improvement of efficiency comparing it with the one obtained before:

$$e_2 = \frac{L_{mt}}{L_2} = 0.923\,.$$

Therefore the answer we propose for the minimal path of the satellite is:

$$2L_2 = 75.0R = 130.3 \cdot 10^3\,km\,.$$

There was a natural decision to organise a joint seminar between the team of "Dini" and the researchers of Pisa University aiming at finding the complete solution of the problem.

4 Conclusions

We have to accept the "knowledge that we know nothing" (Apology of Socrates 23b, 29b). The limitations of our skills and possibilities to understand the world around us should be a deep motivation for further attempts to pass over the limits.

References

[1] **http://www.dm.unipi.it/~eroe/index.php** , site of the "Galilei" competition

[2] **http://www.dm.unipi.it/~eroe/problemi.php** , problems proposed by Math Labs

[3] **http://www.nasa.gov**