Travelling the space at 1 g acceleration

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1 Introduction

It is obvious that the civilization will have to get to raw materials in the outer space in relative short time otherwise it will suffer from the lack of that. The scenario is known: a rocket engine will accelerate the spacecraft up to the travelling velocity, then the engine will be switched off, and the crew will travel at zero gravity until they reach the destination and decelerate to the landing velocity. These expeditions are supposed to take long time, e.g. Earth − Mars − Earth expedition is estimated to take 520 days [1]. Such a long state of weightlessness might have a devastating impact on the health of the crew. However, there is a possibility to avoid that − accelerate and decelerate the ship at 1 g, which would simulate the Earth's gravity. In this article, we investigate such travelling at Earth – Moon, Earth – Mars, and Earth – Pluto distances. The main questions are how long would take the journeys, and what the maximum speed at the journeys would be. These can easily be answered using first year secondary school physics, and the results are very interesting – they suggest that the exploitation of the planets may start. However, they tell us nothing about the spaceship: What are the properties of the engine? What are the energy requirements? Are these journeys possible at all? That is why we have to go further, and investigate the properties of the spaceship and the rocket engine from the aspect of exhaust velocity and energy requirement. That problem belongs to dynamics of bodies with variable mass, and needs applying calculus. The topic is aimed for students who are interested in rocketry and space technology, and who have acquired the basic of calculus.

Fig.1 Space shuttle launch

2 Curriculum items covered by this unit

2.1 Mathematical curriculum

- Manipulation of terms containing fractions and exponentials
- Derivatives, integral calculus
- Solving simple differential equations

2.2 Science curriculum

- Uniformly accelerated motion
- Mechanical work
- Energy conversion

3 Tasks and problems

3.1 Journey duration and maximum reached speed

We suppose a spaceship moving along a line. It starts at zero speed, accelerates at 1 g halfway, then switches the engine off, turns about at 180° (the necessary time is as short as possible; we omit that), switches the engine on, and decelerates at 1 g until zero speed. Let the total time of the journey be t_T. The acceleration takes $t_T/2$, hence the reached velocity (the maximum velocity of the journey) and distance are

$$v_T = g\frac{t_T}{2},\; d_T = \frac{1}{2}g\left(\frac{t_T}{2}\right)^2. \tag{1}$$

The deceleration takes $t_T/2$ again, thus the done distance is the same. The entire distance is

$$d_T = \frac{1}{2}g\left(\frac{t_T}{2}\right)^2 + \frac{1}{2}g\left(\frac{t_T}{2}\right)^2 = \frac{g{t_T}^2}{4}. \tag{2}$$

Then

$$t_T = 2\sqrt{\frac{d_T}{g}}, \tag{3}$$

and consequently

$$v_T = \sqrt{g d_T}. \tag{4}$$

We get $t_T \approx 3.5$ hours, $v_T \approx 61$ km/s at Earth – Moon distance 3.8×10^5 km [2], $t_T \approx 2$ days, $v_T \approx 858$ km/s at Earth – Mars distance 7.5×10^7 km [3], and $t_T \approx 18$ days, $v_T \approx 7{,}542$ km/s at Earth – Pluto distance 5.8×10^9 km [4].

The values of total time t_T are interesting. They suggest that the exploitation of the planets may start. However, we know nothing about the spaceship – what the properties of the rocket engine are, and what the energy requirements are.

3.2 Tsiolkovsky rocket equation

The thrust (i.e. the force) of an ideal rocket engine is given by the equation [5]

$$F = v_v m_s, \tag{5}$$

where m_s is the propellant mass flow rate (i.e. the mass of the fuel that is burned up per second and that runs off the nozzle as a hot gas), and v_v is the exhaust speed (i.e. the speed of the gas relatively to the spaceship). Let v_v be constant. Let m_s be a function of time. Let *M* be the initial spacecraft mass at $t = 0$. The mass *m* of the spacecraft at time *t* is then

$$m = M - \int_0^t m_s(z)\,\mathrm{d}z. \tag{6}$$

Newton's law gives (acceleration is 1 g)

$$v_{\mathrm{v}} m_{\mathrm{s}} = g\left(M - \int_0^t m_{\mathrm{s}}\,\mathrm{d}z\right). \tag{7}$$

Using differentiating by time, we get

$$v_{\mathrm{v}} \frac{\mathrm{d}m_{\mathrm{s}}}{\mathrm{d}t} = -g m_{\mathrm{s}}. \tag{8}$$

Separation of variables and integration gives

$$m_{\mathrm{s}} = C\,\mathrm{e}^{-\frac{g}{v_{\mathrm{v}}}t}, \tag{9}$$

where C is a constant. Let the propellant mass flow rate be M_{s} at $t = 0$. Then $C = M_{\mathrm{s}}$, and

$$m_{\mathrm{s}} = M_{\mathrm{s}}\,\mathrm{e}^{-\frac{g}{v_{\mathrm{v}}}t}. \tag{10}$$

If $t = 0$, then Eq. (5) gives

$$gM = v_{\mathrm{v}} M_{\mathrm{s}}, \tag{11}$$

from witch

$$M_{\mathrm{s}} = \frac{gM}{v_{\mathrm{v}}}, \tag{12}$$

and finally

$$m_{\mathrm{s}} = \frac{gM}{v_{\mathrm{v}}}\mathrm{e}^{-\frac{g}{v_{\mathrm{v}}}t}. \tag{13}$$

The time of combustion equals t_{T}, thus the total mass m_{f} of the fuel is

$$m_{\mathrm{f}} = \int_0^{t_{\mathrm{T}}} m_{\mathrm{s}}\,\mathrm{d}t = \frac{gM}{v_{\mathrm{v}}}\int_0^{t_{\mathrm{T}}} \mathrm{e}^{-\frac{g}{v_{\mathrm{v}}}t}\,\mathrm{d}t = -M\left[\mathrm{e}^{-\frac{g}{v_{\mathrm{v}}}t}\right]_0^{t_{\mathrm{T}}} = M\left(1 - \mathrm{e}^{-\frac{g}{v_{\mathrm{v}}}t_{\mathrm{T}}}\right). \tag{14}$$

The relative mass m_{fr} of the fuel at the start is

$$m_{\mathrm{fr}} = \frac{m_{\mathrm{f}}}{M} = 1 - \mathrm{e}^{-\frac{g}{v_{\mathrm{v}}}t_{\mathrm{T}}}, \tag{15}$$

from which it follows that

$$t_{\mathrm{T}} = -\frac{v_{\mathrm{v}}}{g}\ln\left(1 - m_{\mathrm{fr}}\right). \tag{16}$$

We note:

1) $0 < \left(1 - m_{\mathrm{fr}}\right) < 1$, then $\ln\left(1 - m_{\mathrm{fr}}\right) < 0$, and $t_{\mathrm{T}} > 0$.

2) If the spaceship accelerated all the time up to t_{T}, the terminal velocity would be

$$v_{\mathrm{T}} = g t_{\mathrm{T}} = -v_{\mathrm{v}}\ln\left(1 - m_{\mathrm{fr}}\right). \tag{17}$$

Rearranging Eq. (17) and introducing M_0 for the mass of the spaceship without the fuel yield the fundamental equation of rocketry, the famous Tsiolkovsky equation [6]

$$v_{\mathrm{T}} = -v_{\mathrm{v}} \ln(1 - m_{\mathrm{fr}}) = -v_{\mathrm{v}} \ln\left(1 - \frac{m_{\mathrm{f}}}{M}\right) = -v_{\mathrm{v}} \ln\left(\frac{M - m_{\mathrm{f}}}{M}\right) = -v_{\mathrm{v}} \ln\left(\frac{M_0}{M}\right) = v_{\mathrm{v}} \ln\left(\frac{M}{M_0}\right). \tag{18}$$

3.3 Rocket engine minimum exhaust velocity

The spaceship in our model accelerates up to $t_{\mathrm{T}}/2$. The maximum speed is (see Eq. (1))

$$v_{\mathrm{T}} = g\frac{t_{\mathrm{T}}}{2} = -\frac{v_{\mathrm{v}}}{2}\ln(1 - m_{\mathrm{fr}}) \tag{19}$$

Then, the spacecraft decelerates. The entire distance done at working engine is (see Eq. (2))

$$d_{\mathrm{T}} = \frac{g{t_{\mathrm{T}}}^2}{4} = \frac{v_{\mathrm{v}}^2}{4g}\ln^2(1 - m_{\mathrm{fr}}). \tag{20}$$

Consequently

$$v_{\mathrm{v}} = -2\frac{\sqrt{gd_{\mathrm{T}}}}{\ln(1 - m_{\mathrm{fr}})}. \tag{21}$$

Minimum exhaust velocity of the rocket engine at various fuel relative mass is in Table 1.

m_{fr}	v_v (km/s)		
(%)	Earth – Moon	Earth – Mars	Earth – Pluto
99	27	370	3,300
90	53	750	6,600
50	180	2,500	22,000

Table 1 Rocket engine minimum exhaust velocity

Current rocket engines have maximum $v_{\mathrm{v}} \approx 4.4$ km/s [7]. At $m_{\mathrm{fr}} = 90$ %, Eqs. (16, 19, 20) give $t_{\mathrm{T}} = 1{,}000$ s, $v_{\mathrm{T}} \approx 5$ km/s, $d_{\mathrm{T}} \approx 2{,}700$ km. The distance is too short to make the way of space travel meaningful. Hence, 1g space travel is a topic of the future when technically new rocket engines [8, 9] take place.

3.4 Rocket engine energy requirement

The work done by the engine during combustion time t_{T} is

$$W = \int_0^s \mathbf{F}\mathbf{ds} = \int_0^{t_{\mathrm{T}}} \mathbf{F}\mathbf{v}\,\mathrm{d}t = \int_0^{t_{\mathrm{T}}} Fv\,\mathrm{d}t \tag{22}$$

The thrust of the engine is given by Eqs. (5, 13), the velocity of the ship is $v = gt$, hence

$$W = g^2 M \int_0^{t_{\mathrm{T}}} t\,\mathrm{e}^{-\frac{g}{v_{\mathrm{v}}}t}\,\mathrm{d}t = Mv_{\mathrm{v}}^2\left[1 - \left(1 + \frac{gt_{\mathrm{T}}}{v_{\mathrm{v}}}\right)\mathrm{e}^{-\frac{gt_{\mathrm{T}}}{v_{\mathrm{v}}}}\right]. \tag{23}$$

The total mass m_{f} of the fuel is given by Eq. (14). The required energy for a unit mass of the fuel i.e. the fuel energy density is then

$$W_{\mathrm{fu}} = \frac{W}{m_{\mathrm{f}}} = v_{\mathrm{v}}^2\frac{1 - \left(1 + \frac{gt_{\mathrm{T}}}{v_{\mathrm{v}}}\right)\mathrm{e}^{-\frac{gt_{\mathrm{T}}}{v_{\mathrm{v}}}}}{1 - \mathrm{e}^{-\frac{g}{v_{\mathrm{v}}}t_{\mathrm{T}}}} = 4gd_{\mathrm{T}}\frac{m_{\mathrm{fr}} + (1 - m_{\mathrm{fr}})\ln(1 - m_{\mathrm{fr}})}{m_{\mathrm{fr}}\ln^2(1 - m_{\mathrm{fr}})}. \tag{24}$$

Fuel minimum energy density at various fuel relative mass is in Table 2.

m_{fr} (%)	W_{fu} (J/kg)		
	Earth – Moon	**Earth – Mars**	**Earth – Pluto**
99	6.7×10^8	1.3×10^{11}	1.0×10^{13}
90	2.1×10^9	4.1×10^{11}	3.2×10^{13}
50	9.5×10^9	1.9×10^{12}	1.5×10^{14}

Table 2 Fuel minimum energy density

Petrol has the maximum calorific value (i.e. energy density) of all fossil fuels, and it is $W_{fu} \approx 4.7 \times 10^7$ J/kg [10]. Liquid hydrogen has the maximum calorific value of all combustible fuels, and it is $W_{fu} \approx 1.4 \times 10^8$ J/kg. The energy density of fissile fuel U 235 is $W_{fu} \approx 7.7 \times 10^{13}$ J/kg. The energy density of hydrogen as a fusion fuel is $W_{fu} \approx 3.0 \times 10^{14}$ J/kg.

We can see that only fission and fusion engines make the journeys possible. Finally, formula $E = mc^2$ gives $W_{fu} \approx 9 \times 10^{16}$ J/kg. Then, Eq. (24) gives $m_{fr} \approx 0.13\,\%$ at Earth – Pluto distance, which is the relative mass of about 2 litres of petrol in a car. Hence, 1g interplanetary transport by matter annihilator propelled spaceship is going to be very effective.

3.5 Conclusions

The preconditions for accomplishing Earth − Moon, Earth – Mars, and Earth – Pluto journeys at 1 g acceleration (halfway +1g, halfway −1g) if the fuel takes 50 % of the starting mass of the ship are:

1) rocket engine exhaust velocity of 180 km/s, 2500 km/s, and 22500 km/s;
2) fuel energy density of 10 GJ/kg, 2 TJ/kg, and 150 TJ/ kg, which fission or fusion fuel can fulfil only.

References

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